International Journal of Theoretical Physics, Vol. 33, No. 4, 1994

Nonstandard Fock Spaces

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Received April 23, 1993

A theory of nonstandard inner product spaces is developed using methods of nonstandard analysis. Various results concerning nonstandard operators and their spectra are proved. The theory is applied to construct nonstandard Fock spaces which extend the standard Fock spaces. Moreover, a rigorous framework for the field operators of quantum field theory is presented. The results are illustrated for the case of Klein–Gordon fields.

1. INTRODUCTION

Although quantum field theory based on Fock space has attained various numerical and theoretical successes, its lack of mathematical rigor can lead to inconsistencies. This lack of rigor also inhibits our deep understanding of the theory and may possibly produce an impediment to further progress. Standard Fock space quantum field theory contains certain mathematically ill-defined concepts which must be remedied to make it rigorous. The most important of these are delta functions and other "generalized functions," "unnormalizable" plane waves, and the manipulation of infinities. We do not advocate removing these concepts, since we would lose the predictive and numerical power of the theory. However, we do advocate that these concepts be retained in a rigorous reformulation.

In this paper, we shall extend the standard Fock space to a nonstandard space that is large enough to contain the "generalized functions" and also allows one to manipulate infinities algebraically in a rigorous fashion. The construction of the enlarged Fock space relies on the theory of nonstandard analysis. Our main intention is to develop the properties of a

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0020-7748/94/0400-0865$07.00/0

nonstandard Fock space and the corresponding nonstandard field operators. We have not yet developed a theory of interacting fields, but we shall illustrate our results for the case of free Klein–Gordon fields. The framework that we shall present has formal similarities to the standard theory. This is as it should be, since we do not want to lose the power of standard quantum field theory.

We are not the first to apply nonstandard analysis to the problems of quantum theory. There is already a considerable amount of literature devoted to this subject (Albeverio *et al.*, 1986; Farrukh, 1975; Francis, 1981; Kelemen and Robinson, 1972; Nakamuro, 1991; Thurber and Katz, 1974; Todorov, 1985). However, most previous investigations have applied the nonstandard theory to obtain results about the standard universe. In this work we contend that the nonstandard universe has physical significance in its own right. For example, we propose that the states and observables of nonstandard Fock space have physical content.

This paper is organized as follows. Since the reader may not be familiar with nonstandard analysis, we present a simplified overview in Section 2. This section stresses only the concepts of nonstandard analysis that will be needed for the sequel. Section 3 develops the theory of hyper-inner product spaces and operators on such spaces. Emphasis is placed on the boundedness and spectral properties of these operators. Section 4 specializes the work in Section 3 to investigate internal inner product spaces and internal operators. Nonrelativistic nonstandard Fock spaces are constructed in Section 5 and their corresponding field operators are developed in Section 6. Second quantization of operators and its relationship to field operators are discussed in Section 7. A relativistic nonstandard Fock space is constructed in Section 8 and the theory is illustrated for Klein–Gordon fields.

2. NONSTANDARD ANALYSIS

This section briefly reviews the theory of nonstandard analysis (Davis, 1977; Hurd and Loeb, 1985; Manchover and Hirschfeld, 1969; Robinson, 1966; Strogan and Luxemburg, 1976). Our presentation follows Lindstrøm (1988) fairly closely and employs the ultrapower construction for our nonstandard model. Although the transfer principle is one of the basic tools of nonstandard analysis, we shall not use it, for two reasons. First, the transfer principle relies fairly heavily on the machinery of logic. Second, although this principle is quite powerful, its abstraction hides the constructive and analytic properties of the model. We begin by constructing the nonstandard complex field $^*\mathbb{C}$.

Let $\mathbb{N}$ be the set of nonnegative integers and let $2^{\mathbb{N}}$ be the power set of $\mathbb{N}$. Let m be a nonatomic, 0–1, finitely additive measure on $2^{\mathbb{N}}$. That is, m

is a mapping m: $2^{\mathbb{N}} \to \{0, 1\}$ satisfying:

(1) $m(\{a\}) = 0$ for every $a \in \mathbb{N}$.
(2) $m(\mathbb{N}) = 1$.
(3) $m(A \cup B) = m(A) + m(B)$ for every $A, B \in 2^{\mathbb{N}}$ with $A \cap B = \emptyset$.

It follows that m vanishes on every finite set and is unity on every cofinite set. There is a natural one-to-one correspondence between such measures and free ultrafilters on $2^{\mathbb{N}}$. A simple Zorn's lemma argument implies the existence of free ultrafilters on $2^{\mathbb{N}}$ and hence such measures must exist. Although m is not unique, it is irrelevant which m we choose and we shall work with one fixed m in the sequel.

The following lemma summarizes the properties of m that we shall need. These properties are well known and the simple proofs are omitted. We denote the complement of a set A by A'.

Lemma 2.1. For every $A, B \in 2^{\mathbb{N}}$, we have:

(a) $m(A) = 1$ or $m(A') = 1$ (but not both).
(b) If $A \subseteq B$, then $m(A) \leq m(B)$.
(c) If $m(A) = m(B) = 1$, then $m(A \cap B) = 1$.

Let $s = \mathbb{C}^{\mathbb{N}}$ be the set of all complex-valued sequences. For (a_n), $(b_n) \in s$ we define $(a_n) \sim (b_n)$ if

$$m(n \in \mathbb{N}: a_n = b_n\} = 1$$

in which case we write $a_n = b_n$ a.e. It is easy to show that $\sim$ is an equivalence relation. For example, to prove transitivity, suppose $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n)$. Let

$$A = \{n \in \mathbb{N}: a_n = b_n\}$$

$$B = \{n \in \mathbb{N}: b_n = c_n\}$$

$$C = \{n \in \mathbb{N}: a_n = c_n\}$$

Then $m(A) = m(B) = 1$, so by Lemma 2.1(c), $m(A \cap B) = 1$. Since $A \cap B \subseteq C$, applying Lemma 2.1(b), we have $m(C) = 1$. Hence, $(a_n) \sim (c_n)$.

We define the *hypercomplex numbers* $^*\mathbb{C}$ by $^*\mathbb{C} = s/\sim$. Denoting the equivalence class containing (a_n) by $[a_n]$, we have

$$^*\mathbb{C} = \{[a_n]: (a_n) \in s\}$$

As in the above proof of transitivity, if $(a_n) \sim (b_n)$ and $a_n \in \mathbb{R}$ a.e., then $b_n \in \mathbb{R}$ a.e. We define the *hyperreals* $^*\mathbb{R}$ by

$$^*\mathbb{R} = \{[a_n] \in {}^*\mathbb{C}: a_n \in \mathbb{R} \text{ a.e.}\}$$

and our previous observation shows that $^*\mathbb{R}$ is well-defined. Moreover, $^*\mathbb{R} \subseteq {}^*\mathbb{C}$. We define addition and multiplication on $^*\mathbb{C}$ by

$$[a_n] + [b_n] = [a_n + b_n]$$
$$[a_n][b_n] = [a_n b_n]$$

and order on $^*\mathbb{R}$ by

$$[a_n] \le [b_n] \qquad \text{if} \quad a_n \le b_n \text{ a.e.}$$

As in our previous proof of transitivity, it is easy to show that these are well-defined. For $a \in \mathbb{C}$, we define the sequence $(\bar{a})$ by $\bar{a}_n = a$ for all $n \in \mathbb{N}$. The *zero* and *unit* of $^*\mathbb{C}$ are defined as $0 = [\bar{0}]$ and $1 = [\bar{1}]$, respectively.

Theorem 2.2. $^*\mathbb{C}$ is a field and $^*\mathbb{R} \subseteq {}^*\mathbb{C}$ is an ordered subfield.

Proof. There are many straightforward verifications to be made and we shall demonstrate two of them. To show the existence of inverses, suppose $[a_n] \neq 0$. Letting $A = \{n \in \mathbb{N}: a_n \neq 0\}$, we have $m(A) = 1$. For $n \in A$, let $b_n = 1/a_n$ and for $n \in A'$, let $b_n = 0$. Then

$$[a_n][b_n] = [a_n b_n] = [\bar{1}] = 1$$

To show dichotomy of order on $^*\mathbb{R}$, let $[a_n], [b_n] \in {}^*\mathbb{R}$. Let $A = \{n \in \mathbb{N}: a_n \le b_n\}$. If $m(A) = 1$, then $a_n \le b_n$ a.e., so $[a_n] \le [b_n]$. If $m(A) = 0$, then by Lemma 2.1(a), $m(A') = 1$. Since $A' = \{n \in \mathbb{N}: b_n < a_n\}$, we have $b_n \le a_n$ a.e., so $[b_n] \le [a_n]$. Similar proofs apply to the other properties. ∎

The injection $a \mapsto [\bar{a}]$ imbeds $\mathbb{R}$ into $^*\mathbb{R}$ and $\mathbb{C}$ into $^*\mathbb{C}$. It is straightforward to show that this mapping is a field isomorphism from $\mathbb{C}$ onto its range and an ordered field isomorphism from $\mathbb{R}$ onto its range. Hence, by identification, we can and will assume that $\mathbb{R} \subseteq {}^*\mathbb{R}$ and $\mathbb{C} \subseteq {}^*\mathbb{C}$. We call elements of $\mathbb{C}$ *standard* and elements of $^*\mathbb{C} \backslash \mathbb{C}$ *nonstandard.*

For $[a_n] \in {}^*\mathbb{C}$ we define the *modulus* $|[a_n]| = [|a_n|]$. This mapping is well-defined and has the usual properties of a modulus. In a similar way, we define the *complex conjugate* $[a_n]^* = [a_n^*]$ and again this operation has the usual properties. We also define the real and imaginary parts in the traditional way. An element $x \in {}^*\mathbb{C}$ is *infinitesimal* if $|x| < \varepsilon$ for every $\varepsilon \in \mathbb{R}^+ = \{a \in \mathbb{R}: a > 0\}$. An element $x \in {}^*\mathbb{C}$ is *finite* if $|x| < a$ for some $a \in \mathbb{R}^+$. If $x \in {}^*\mathbb{C}$ is not finite, it is *infinite.*

Notice that 0 is the only standard infinitesimal. The element $\delta_1 = [1/(n+1)]$ is infinitesimal since for every $\varepsilon \in \mathbb{R}^+$, the set

$$A = \{n \in \mathbb{N}: 1/(n+1) < \varepsilon\}$$

contains all but a finite number of n's so $m(A) = 1$. Since $\delta_1 > 0$, δ_1 is a nonstandard infinitesimal. Also, $\delta_2 = [1/(n+1)^2]$ is infinitesimal and

$0 < \delta_2 < \delta_1$. Moreover, $[n]$ and $[n^2]$ are infinite. These examples show that $^*\mathbb{R}$ and $^*\mathbb{C}$ are proper field extensions of $\mathbb{R}$ and $\mathbb{C}$, respectively. Moreover, since $^*\mathbb{R}$ is an ordered field (and $^*\mathbb{C}$ is a field), infinite numbers can be algebraically manipulated and compared. The proof of the next theorem is essentially that given in Hurd and Loeb (1985) and Lindstrøm (1988).

Theorem 2.3. Any finite $x \in {}^*\mathbb{C}$ can be uniquely represented as a sum $x = a + \varepsilon$, where $a \in \mathbb{C}$ and ε is infinitesimal.

For $x, y \in {}^*\mathbb{C}$, if $x - y$ is infinitesimal, we write $x \approx y$ and say that x and y are *close*. It is easy to show that $\approx$ is an equivalence relation. For any finite $x \in {}^*\mathbb{C}$, the unique $a \in \mathbb{C}$ such that $x \approx a$ is the *standard part* of x and is denoted $a = {}^\circ x$ or $a = \mathrm{st}(x)$. We consider st as a mapping from the set of finite elements $\mathrm{Fin}({}^*\mathbb{C})$ of $^*\mathbb{C}$ onto $\mathbb{C}$. It is easy to show that st preserves addition and multiplication on $\mathrm{Fin}({}^*\mathbb{C})$ and moreover preserves $\leq$ on $\mathrm{Fin}({}^*\mathbb{R})$. The *monad* of $a \in \mathbb{C}$ is

$$\mathrm{Mon}(a) = \{x \in {}^*\mathbb{C} \colon x \approx a\} = \{x \in {}^*\mathbb{C} \colon {}^\circ x = a\}$$

Thus, Mon(0) is the set of infinitesimals.

Lemma 2.4. If $a_n \in \mathbb{C}$ and $\lim a_n = a$, then $[a_n] \approx a$.

Proof. For any $\varepsilon \in \mathbb{R}^+$, $A = \{n \in \mathbb{N} \colon |a_n - a| < \varepsilon\}$ contains all but a finite number of n's so $m(A) = 1$. Hence, $|[a_n] - a| < \varepsilon$. ■

The converse of Lemma 2.4 does not hold. For example, either $[1, 0, 1, 0, \ldots]$ or $[0, 1, 0, 1, \ldots]$ equals 0.

We now show how to transfer structures from $\mathbb{C}$ to $^*\mathbb{C}$. If $A \subseteq {}^*\mathbb{C}$, we define $^*A \subseteq {}^*\mathbb{C}$ by

$${}^*A \equiv [\bar{A}] = \{[a_n] \in {}^*\mathbb{C} \colon a_n \in A \text{ a.e.}\}$$

The set $^*A$ is well-defined and subsets of $^*\mathbb{C}$ of the form $^*A$ are *standard*. It is clear that $A \subseteq {}^*A$. Moreover, it is easy to show that $A = {}^*A$ if and only if A has finite cardinality. If $A \subseteq \mathbb{C}$ and $f \colon A \to \mathbb{C}$, we define $^*f \colon {}^*A \to {}^*\mathbb{C}$ by

$${}^*f([a_n]) \equiv [\bar{f}]([a_n]) = [f(a_n)]$$

The function $^*f$ is well-defined and functions of this form are *standard*. It is clear that $^*f$ extends f.

If $A_n \subseteq \mathbb{C}$, $n \in \mathbb{N}$, we define $[A_n] \subseteq {}^*\mathbb{C}$ by $[a_n] \in [A_n]$ if and only if $a_n \in A_n$ a.e. Subsets of $^*\mathbb{C}$ of this form are *internal*. If $A_n \subseteq \mathbb{C}$ and $f_n \colon A_n \to \mathbb{C}$, $n \in \mathbb{N}$, we define $[f_n] \colon [A_n] \to {}^*\mathbb{C}$ by $[f_n]([a_n]) = [f_n(a_n)]$ and such functions are *internal*. It is clear that standard sets and functions are internal, while the converse does not hold. Sets and functions that are not internal are *external.* An example of an internal set that is not standard is

$$[a, b] = \{x \in {}^*\mathbb{R}: a \le x \le b\}$$

where $a \le b$, $a, b \in {}^*\mathbb{R}\backslash\mathbb{R}$. Examples of external sets are Mon(0), $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$.

We can transfer operations on sets and functions in $\mathbb{C}$ to internal sets and functions. For example, let $A_n \subseteq \mathbb{R}$ be measurable sets and let $f_n: A_n \to \mathbb{C}$ be integrable functions. If $A = [A_n]$ and $f = [f_n]$, we define the *internal integral*

$$\int_A f\,dx = \left[\int_{A_n} f_n\,dx\right] \in {}^*\mathbb{C}$$

This integral inherits many of the standard properties.

We now show that delta functions and other generalized functions are well-defined internal functions on $^*\mathbb{R}$. For example, let $\varepsilon = [1/n]$, where we assume the zeroth term is 1, and define the internal function $\delta_0: {}^*\mathbb{R} \to {}^*\mathbb{R}$ by

$$\delta_0(x) = (2\pi\varepsilon)^{-1/2} e^{-x^2/2\varepsilon} = \left[\left(\frac{n}{2\pi}\right)^{1/2} e^{-nx^2/2}\right]$$

If $f: \mathbb{R} \to \mathbb{C}$ is continuous and decreases rapidly, then

$$\int \delta_0(x) f(x)\,dx = \left[\left(\frac{n}{2\pi}\right)^{1/2} \int e^{-nx^2/2} f(x)\,dx\right]$$

It is well known that this sequence converges to $f(0)$, so by Lemma 2.4 we have

$$\int \delta_0(x) f(x)\,dx \approx f(0)$$

In other words,

$$\mathrm{st} \int \delta_0(x) f(x)\,dx = f(0)$$

The method that we have used to define $^*\mathbb{C}$ is called the *ultrapower construction.* This construction can be applied to any set $\mathscr{S}$. We define $^*\mathscr{S}$ as the set of equivalence classes of sequences in $\mathscr{S}^{\mathbb{N}}$. In an analogous way we have $\mathscr{S} \subseteq {}^*\mathscr{S}$ and we define internal sets and functions on $^*\mathscr{S}$ as before. The next theorem is called the *saturation principle* and its proof can be found in Lindstrøm (1988).

Theorem 2.5. Let (A_i) be a sequence of internal sets in $^*\mathscr{S}$. If $\bigcap_{i \le n} A_i \ne \emptyset$ for every $n \in \mathbb{N}$, then $\bigcap_{i \in \mathbb{N}} A_i \ne \emptyset$.

It is easy to see that the family of internal sets in $^*\mathscr{S}$ is closed under finite Boolean operations and hence forms an algebra. Indeed, we have

$$[A_n] \cap [B_n] = [A_n \cap B_n]$$
$$[A_n] \cup [B_n] = [A_n \cup B_n]$$
$$[A_n]' = [A_n']$$

However, this algebra is as far from being a σ-algebra as it could possibly be.

Corollary 2.6. If (A_i) is a sequence of internal sets in $^*\mathscr{S}$, then $\bigcup_{i \in \mathbb{N}} A_i$ is internal if and only if $\bigcup_{i \in \mathbb{N}} A_i = \bigcup_{i \leq n} A_i$ for some $n \in \mathbb{N}$.

Proof. Sufficiency is trivial. For necessity, suppose $A = \bigcup_{i \in \mathbb{N}} A_i$ is internal. Then $A \backslash A_i$ is internal for all $i \in \mathbb{N}$ and clearly $\bigcap_{i \in \mathbb{N}} (A \backslash A_i) = \emptyset$. By the saturation principle, there exists an $n \in \mathbb{N}$ such that $\bigcap_{i \leq n} (A \backslash A_i) = \emptyset$, so $A = \bigcup_{i \leq n} A_i$. ■

We now discuss internal measure theory. Let $\mathscr{S}$ be a set and let $\Omega = [\Omega_n]$, $\Omega_n \subseteq \mathscr{S}$, be an internal subset of $^*\mathscr{S}$. Let $\mathscr{A}_n$ be a σ-algebra on Ω_n, $n \in \mathbb{N}$. We define the *internal algebra* $\mathscr{A} = [\mathscr{A}_n]$ on Ω as follows. A set $A \in \mathscr{A}$ if and only if $A = [A_n]$ is an internal set with $A_n \in \mathscr{A}_n$ a.e. As we have seen, $\mathscr{A}$ is indeed an algebra, but it is not a σ-algebra except in trivial cases. Let $\hat{\mathscr{A}}$ be the σ-algebra generated by $\mathscr{A}$. Let μ_n be a measure on $\mathscr{A}_n$, $n \in \mathbb{N}$. Then $\mu = [\mu_n]$ is defined on $\mathscr{A}$ by

$$\mu(A) = [\mu_n]([A_n]) = [\mu_n(A_n)] \in {}^*\mathbb{R} \cup \{\infty\}$$

Now it is easy to check that μ is finally additive on $\mathscr{A}$. Define the finitely additive measure $^\circ\mu\colon \mathscr{A} \to \mathbb{R} \cup [\infty\}$ by

$$^\circ\mu(A) = \begin{cases} ^\circ(\mu(A)) & \text{if } \mu(A) \text{ is finite} \\ \infty & \text{if } \mu(A) \text{ is finite or } \infty \end{cases}$$

If $A_i \in \mathscr{A}$, $i \in \mathbb{N}$, are mutually disjoint and $\bigcup_{i \in \mathbb{N}} A_i \in \mathscr{A}$, then by the saturation principle, $A_i = \emptyset$ except for a finite number of i's. Hence, $^\circ\mu$ is σ-additive on $\mathscr{A}$. By the Hahn extension theorem (Dunford and Schwartz, 1958; Reed and Simons, 1972), $^\circ\mu$ has a σ-additive extension $(^\circ\mu)^\wedge$ to $\hat{\mathscr{A}}$ (the extension is unique if $^\circ\mu$ is σ-finite). Let $L(\mathscr{A})$ be the completion of $\hat{\mathscr{A}}$ relative to $(^\circ\mu)^\wedge$ and let $L(\mu)$ be the completion of $(^\circ\mu)^\wedge$ on $L(\mathscr{A})$. We call $L(\mu)$ the *Loeb measure* of μ and $(\Omega, L(\mathscr{A}), L(\mu))$ a *Loeb measure space* (Hurd and Loeb, 1985; Lindstrøm, 1988).

Let $f\colon \Omega \to {}^*\mathbb{R}$ be internal, where $f = [f_n]$ and $f_n\colon \Omega_n \to \mathbb{R}$ are μ_n-integrable. It is easy to check that $^\circ f$ is $L(\mathscr{A})$-measurable and $L(\mu)$-integrable.

We now have two natural integrals; the *internal integral*

$$\int f\,d\mu = \left[\int f_n\,d\mu_n\right]$$

and the *Loeb integral* $\int {}^\circ f\,dL(\mu)$. We say that f is *finite* if $L(\mu)(\{\omega\in\Omega: f(\omega)\neq 0\})$ and $\sup|f(\omega)|$ are finite. The next theorem is proved in Lindstrøm (1988).

Theorem 2.7. If f is finite, then ${}^\circ\int f\,d\mu = \int {}^\circ f\,dL(\mu)$.

3. HYPER-INNER PRODUCT SPACES

A *hyper-inner product space* $\mathscr{H}$ is an inner product space over the hypercomplex field $^*\mathbb{C}$. That is, $\mathscr{H}$ is a vector space over $^*\mathbb{C}$ with a mapping $\langle\cdot,\cdot\rangle: \mathscr{H}\times\mathscr{H}\to{}^*\mathbb{C}$ satisfying:

(1) $\langle\cdot,\cdot\rangle$ is linear in the second argument.
(2) $\langle x, y\rangle = \langle y, x\rangle^*$ for all $x, y\in\mathscr{H}$.
(3) $\langle x, x\rangle > 0$ if $x\neq 0$.

The *norm* of $x\in\mathscr{H}$ is $\|x\| = \langle x, x\rangle^{1/2}$. We say that $x\in\mathscr{H}$ is *infinitesimal* (*finite*, *infinite*) if $\|x\|$ is infinitesimal (finite, infinite). Moreover, we say that $x, y\in\mathscr{H}$ are *close* and write $x\approx y$ if $\|x-y\|$ is infinitesimal. An example of a hyper-inner product space is $^*\mathbb{C}$ with inner product $\langle a, b\rangle = a^*b$. Although the proof of the following theorem is essentially the same as in the standard case, we include it for completeness.

Theorem 3.1. If x, y are vectors in a hyper-inner product space, then:

(a) $|\langle x, y\rangle| \leq \|x\|\cdot\|y\|$ (Schwarz's inequality).
(b) $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality).

Proof. (a) If $y=0$, then the result is immediate, so assume that $y\neq 0$. Then for any $a\in{}^*\mathbb{C}$, we have

$$\begin{aligned}0 \leq \|x-ay\|^2 &= \langle x-ay, x-ay\rangle\\ &= \|x\|^2 - a^*\langle y, x\rangle - a\langle x, y\rangle + |a|^2\|y\|^2\end{aligned}$$

Setting $a = \langle y, x\rangle/\|y\|^2$, we obtain

$$0 \leq \|x\|^2 - \frac{|\langle y, x\rangle|^2}{\|y\|^2}$$

and the result follows.

(b) Applying (a), we have

$$\begin{aligned}\|x+y\|^2 &= \|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2 \\ &\le \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \\ &\le \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2\end{aligned}$$

and the result follows. ■

Another easily proved property is $\|ax\| = |a| \cdot \|x\|$ for every $a \in {}^*\mathbb{C}$, $x \in \mathscr{H}$. The proof of the next theorem is again essentially the same as in the standard case and we omit it.

Theorem 3.2. If x, y are vectors in a hyper-inner product space, then:

(a) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ (parallelogram law).
(b) $|\langle x, y\rangle| = \|x\| \cdot \|y\|$ if and only if one of the vectors x or y is a scalar multiple of the other.
(c) $|\|x\| - \|y\|| \le \|x-y\|$.
(d) $\langle x, y\rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \|x + i^k y\|^2 \qquad (i = \sqrt{-1})$ (polar identity).

We can apply the previous two theorems to obtain the following corollary.

Corollary 3.3. Let x, y be vectors of a hyper-inner product space.

(a) If x is finite and y is infinitesimal, then $\langle x, y\rangle$ is infinitesimal. If x and y are finite, then $\langle x, y\rangle$ is finite.
(b) The relation $\approx$ is an equivalence relation.
(c) If $x \approx y$, then $\|x\| \approx \|y\|$.
(d) If x is finite and $a \approx a' \in {}^*\mathbb{C}$, then $ax \approx a'x$.
(e) If x and y are finite, then $x + y$ is finite.
(f) If $x \approx y$ and z is finite, then $\langle x, z\rangle \approx \langle y, z\rangle$.

Proof. (a) This follows from Theorem 3.1(a) and the fact that a finite number times an infinitesimal is infinitesimal. The second statement is similar.

(b) Suppose $x \approx y$ and $y \approx z$. Then from Theorem 3.1(b) we have

$$\|x-z\| \le \|x-y\| + \|y-z\| \approx 0$$

(c) From Theorem 3.2(c) we have

$$|\|x\| - \|y\|| \le \|x-y\| \approx 0$$

(d) Since $a \approx a'$, we have

$$\|ax - a'x\| = |a-a'| \cdot \|x\| \approx 0$$

(e) This follows from Theorem 3.1(b).

(f) Applying Theorem 3.1(a) gives

$$|\langle x, z\rangle - \langle y, z\rangle| = |\langle x - y, z\rangle| \leq \|x - y\| \cdot \|z\| \approx 0 \quad \blacksquare$$

If $\varepsilon \in {}^*\mathbb{R}^+$ and $x \in \mathscr{H}$, we define the *ball centered at x of radius* ε by

$$B_\varepsilon(x) = \{y \in \mathscr{H}: \|y - x\| < \varepsilon\}$$

The set of all balls forms a base for a topology on $\mathscr{H}$ called the *norm topology*. Notice that this is a very strong topology since ε can be infinitesimal. We also endow ${}^*\mathbb{C}$ with this same topology. It follows from Theorems 3.1 and 3.2(c) that addition, scalar multiplication, the norm, and the inner product are continuous in the norm topology.

In the norm topology, a hypersequence $x_n \in \mathscr{H}$, $n \in {}^*\mathbb{N}$, *converges to* $x \in \mathscr{H}$ if for any $\varepsilon \in {}^*\mathbb{R}^+$ there exists an $N \in {}^*\mathbb{N}$ such that $\|x_n - x\| < \varepsilon$ for all $n \geq N$. Notice that for any $\varepsilon \in {}^*\mathbb{R}^+$ there exists an $n \in {}^*\mathbb{N}^+$ such that $1/n < \varepsilon$. It follows that the balls $B_{1/n}(x)$, $n \in {}^*\mathbb{N}^+$, $x \in \mathscr{H}$, form a base for the norm topology. Hence, convergence of hypersequences determines the norm topology. That is, a set $A \subseteq \mathscr{H}$ is closed if and only if whenever a hypersequence $x_n \in A$ converges to an $x \in \mathscr{H}$, we have $x \in A$. It follows that a function $f: \mathscr{H} \to \mathscr{H}$ (or $\mathscr{H} \to {}^*\mathbb{C}$) is continuous if and only if for any convergent hypersequence x_n, $n \in {}^*\mathbb{N}$, $x_n \to x$ implies $f(x_n) \to f(x)$. Of course, the norm topology is Hausdorff. In the sequel, when we consider $\mathscr{H}$ as a topological space we shall always assume this is the norm topology.

As usual, a *subspace* of $\mathscr{H}$ is a nonempty subset of $\mathscr{H}$ that is closed under addition and scalar multiplication. If $\mathscr{D} \subseteq \mathscr{H}$ is a subspace, then a mapping $T: \mathscr{D} \to \mathscr{H}$ is linear if

$$T(ax + by) = aTx + bTy$$

for every $x, y \in \mathscr{D}$, $a, b \in {}^*\mathbb{C}$. We call a linear mapping T an *operator* with *domain* $\mathscr{D}$. A subset of $\mathscr{H}$ is *bounded* if it is contained in a ball centered at 0.

Theorem 3.4. If $T: \mathscr{H} \to \mathscr{H}$ is an operator, then the following statements are equivalent. (a) T is continuous. (b) T is continuous at 0. (c) The set $\{\|Tx\|: \|x\| \leq 1\}$ is bounded. (d) There exists an $M \in {}^*\mathbb{R}^+$ such that $\|Tx\| \leq M\|x\|$ for all $x \in \mathscr{H}$.

Proof. (a) $\Rightarrow$ (b) is trivial. Suppose (b) holds. Then there exists a $\delta \in {}^*\mathbb{R}^+$ such that $\|Ty\| < 1$ if $\|y\| < \delta$. If $\|x\| \leq 1$, then $y = \delta x/2$ has norm $\|y\| < \delta$, so

$$\frac{\delta}{2}\|Tx\| = \|Ty\| < 1$$

Hence, $\|Tx\| < 2/\delta$, so (c) holds. Suppose (c) holds. Then there exists an $M \in {}^*\mathbb{R}^+$ such that $\|Ty\| < M$ if $\|y\| \leq 1$. If $x \neq 0$, then $y = x/\|x\|$ satisfies $\|y\| \leq 1$. Hence,

$$\frac{1}{\|x\|}\|Tx\| = \|Ty\| < M$$

We conclude that $\|Tx\| \leq M\|x\|$, so (d) holds. Finally, suppose (d) holds. If the hypersequence x_n, $n \in \mathbb{N}$, converges to x, we have

$$\|Tx_n - Tx\| = \|T(x_n - x)\| \leq M\|x_n - x\|$$

Now given an $\varepsilon \in {}^*\mathbb{R}^+$, $\|x_n - x\| < \varepsilon/M$ eventually, so $\|Tx_n - Tx\| < \varepsilon$ eventually. Hence, T is continuous and (a) holds. ■

If T satisfies condition (d) of Theorem 3.4, we say that T is *bounded* and M is a bound. Thus, an operator $T: \mathscr{H} \to \mathscr{H}$ is continuous if and only if it is bounded. Unlike the standard case, a continuous operator need not have a finite bound. If $L = \sup\{\|Tx\|: \|x\| \leq 1\}$ exists, we say that T is *normable* with *norm* $\|T\| = L$. If T is normable, then T is bounded. Indeed, if $L = \|T\|$, then $\|Ty\| \leq L$ for all $y \in \mathscr{H}$ with $\|y\| \leq 1$. If $x \neq 0$, then $y = x/\|x\|$ satisfies $\|y\| \leq 1$, so

$$\|Tx\| = \|x\| \cdot \|Ty\| \leq L\|x\|$$

and T is bounded. However, if T is bounded, as we shall later see, T need not be normable.

Theorem 3.5. Let $T: \mathscr{H} \to \mathscr{H}$ be an operator and let

$$\mathscr{M} = \{M \in {}^*\mathbb{R}^+: \|Tx\| \leq M\|x\| \text{ for all } x \in \mathscr{H}\}$$

Then T is normable if and only if $\inf(\mathscr{M})$ exists. Moreover, if T is normable, then $\|T\| = \inf(\mathscr{M})$.

Proof. If T is normable, then, as we have just shown, $\|Tx\| \leq \|T\| \cdot \|x\|$ for all $x \in \mathscr{H}$, so $\|T\| \in \mathscr{M}$. Now let $M \in \mathscr{M}$. If $\|x\| \leq 1$, then $\|Tx\| \leq M\|x\| \leq M$. Hence, $\|T\| \leq M$. Thus $\|T\|$ is a lower bound for $\mathscr{M}$, so $\|T\| = \inf(\mathscr{M})$. Conversely, suppose $\inf(\mathscr{M}) = N$ exists. As before, if $M \in \mathscr{M}$ and $\|x\| \leq 1$, then $\|Tx\| \leq M$. Hence, $\|Tx\| \leq N$, so N is an upper bound for the set

$$\mathscr{N} = \{\|Tx\|: x \in \mathscr{H}, \|x\| \leq 1\}$$

Suppose N_0 is an upper bound for $\mathscr{N}$. Then for any $x \neq 0$, $\|Tx/\|x\|\| \leq N_0$ or $\|Tx\| \leq N_0\|x\|$ and this latter inequality also holds for $x = 0$. Therefore, $N_0 \in \mathscr{M}$ so $N \leq N_0$. Hence, $N = \sup(\mathscr{N})$, so T is normable. ■

Examples of hyper-inner product spaces may be constructed as follows. Let $\mathscr{H}$ be a complex inner product space and let $\mathscr{H} = {}^*\mathscr{H}$. Let $S: \mathscr{H} \times \mathscr{H} \to \mathscr{H}$, $P: \mathbb{C} \times \mathscr{H} \to \mathscr{H}$, $I: \mathscr{H} \times \mathscr{H} \to \mathbb{C}$ be defined by $S(x, y) = x + y$, $P(a, x) = ax$, $I(x, y) = \langle x, y\rangle$. Then ${}^*S, {}^*P, {}^*I$ define an addition, scalar multiplication, and inner product on ${}^*\mathscr{K}$ making ${}^*\mathscr{K}$ a hyper-inner product space.

We now give an example of a bounded (and hence continuous) operator which does not have a finite bound and which is not normable. Let ${}^*\mathscr{K}$ be an infinite-dimensional Hilbert space, let $x_n, n \in \mathbb{N}$, be an orthonormal set in ${}^*\mathscr{K}$, and let $\mathscr{H} = \operatorname{span}\{x_n : n \in \mathbb{N}\} \subseteq {}^*\mathscr{K}$. Then $\mathscr{H}$ is a hyper-inner product space. Define $T: \mathscr{H} \to \mathscr{H}$ by $Tx_n = nx_n, n \in \mathbb{N}$, and extend T by linearity. If $x \in \mathscr{H}$, $\|x\| = 1$, then $x = \sum_{i=1}^m a_i x_{n_i}$, $a_i \in {}^*\mathbb{C}$. Since

$$1 = \|x\|^2 = \sum_{i=1}^m |a_i|^2$$

each of the a_i's is finite. Hence,

$$\|Tx\|^2 = \sum_{i=1}^m |a_i n_i|^2$$

is finite. If $M > 0$ is infinite, it follows that $\|Tx\| \le M\|x\|$ for all $x \in \mathscr{H}$. Hence, T is a bounded operator. However, T has no finite bound, since for $n \in \mathbb{N}$, $\|Tx_n\| = n\|x_n\|$. Moreover, T is not normable. Indeed, suppose $M_0 = \sup\{\|Tx\|: \|x\| \le 1\}$ exists. Then M_0 cannot be infinite since $M_0 - 1$ is also infinite and is a smaller bound for T. Also, as we have seen, M_0 cannot be finite.

The next example shows that even if an operator has a finite bound it may not be normable. Let $\mathscr{H}$ and T be as in the previous example, let $\varepsilon > 0$ be infinitesimal, and define $T_1 = \varepsilon T$. If $x \in \mathscr{H}$ with $\|x\| = 1$, then writing x as before, each of the a_i is finite. Hence,

$$\|T_1 x\|^2 = \varepsilon^2 \sum_{i=1}^m |a_i n_i|^2$$

is infinitesimal. If $M > 0$ is not infinitesimal, we have $\|T_1 x\| \le M\|x\|$ for all $x \in \mathscr{H}$. Hence, T_1 is a bounded operator and has finite bounds. However, T_1 is not normable. Indeed, suppose $\delta = \sup\{\|T_1 x\|: \|x\| \le 1\}$ exists. Then δ must be infinitesimal, since otherwise $\delta/2$ is a smaller bound. Suppose δ is infinitesimal. Then for any $n \in \mathbb{N}$

$$n\varepsilon = \|n\varepsilon x_n\| = \|T_1 x_n\| \le \delta$$

Hence, $2n\varepsilon \le \delta$ or $n\varepsilon \le \delta/2$ for every $n \in \mathbb{N}$. It follows that $a\varepsilon \le \delta/2$ for every finite a. If $x \in \mathscr{H}$, $\|x\| = 1$, then writing x as before, since $|a_i| \le 1$, we have

$$\|T_1 x\|^2 = \varepsilon^2 \sum_{i=1}^{m} |a_i|^2 |n_i|^2 \leq \varepsilon^2 \sum_{i=1}^{m} |n_i|^2$$

Hence,

$$\|T_1 x\| \leq \varepsilon \left[\sum_{i=1}^{m} |n_i|^2 \right]^{1/2} \leq \frac{\delta}{2}$$

Thus $\delta/2$ is a smaller bound, which is a contradiction.

Let T be an operator on $\mathscr{H}$ with dense domain $\mathscr{D}(T)$. Let $\mathscr{D}(T^*)$ be the set of $x \in \mathscr{H}$ for which there exists a $y \in \mathscr{H}$ such that $\langle x, Tz \rangle = \langle y, z \rangle$ for all $z \in \mathscr{D}(T)$. Notice that if y exists it is unique since $\mathscr{D}(T)$ is dense. For each such $x \in \mathscr{D}(T^*)$, we define $T^*x = y$. It is easy to show that T^* is linear and hence is an operator with domain $\mathscr{D}(T^*)$. We say that T *is symmetric* if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for every $x, y \in \mathscr{D}(T)$. Hence, if T is symmetric, then T^* is an extension of T. We say that T is *self-adjoint* if T is symmetric and $\mathscr{D}(T) = \mathscr{D}(T^*)$. Thus, T is self-adjoint if and only if $T = T^*$.

For an operator T *on* $\mathscr{H}$, $\lambda \in {}^*\mathbb{C}$ is in the *resolvent set* $\rho(T)$ if $\lambda I - T$ is a bijection of $\mathscr{D}(T)$ onto $\mathscr{H}$ with a bounded inverse. The complement of $\rho(T)$ is the *spectrum* $\sigma(T)$. The *point spectrum* $\sigma_p(T)$ is the set of $\lambda \in {}^*\mathbb{C}$ such that $\lambda I - T$ is not injective. Thus, $\lambda \in \sigma_p(T)$ if and only if there exists an $x \neq 0$, $x \in \mathscr{H}$, such that $Tx = \lambda x$. We call x an *eigenvector* corresponding to the eigenvalue λ. The continuous spectrum $\sigma_c(T)$ is the set of $\lambda \in {}^*\mathbb{C}$ such that $\lambda \notin \sigma_p(T)$, the range $\operatorname{ran}(\lambda I - T)$ is dense in $\mathscr{H}$, but $(\lambda I - T)^{-1}$ is not bounded. The *residual spectrum* is the set of $\lambda \in {}^*\mathbb{C}$ such that $\lambda \notin \sigma_p(T)$ and $\operatorname{ran}(\lambda I - T)$ is not dense.

Theorem 3.6. (a) If $\lambda \in \sigma_c(T)$, then for any $\varepsilon \in {}^*\mathbb{R}^+$ there exists an $x \in \mathscr{D}(T)$ with $\|x\| = 1$ such that $\|Tx - \lambda x\| < \varepsilon$. (b) If $\lambda \in \sigma_c(T)$, then there exists an $x \in \mathscr{D}(T)$ with $\|x\| = 1$ such that $Tx \approx \lambda x$.

Proof. (a) Let $\lambda \in \sigma_c(T)$ and let $\varepsilon \in {}^*\mathbb{R}^+$. Since $(\lambda I - T)^{-1}$ is unbounded, there exists a $y \in \mathscr{H}$ such that

$$\|(\lambda I - T)^{-1} y\| > \frac{1}{\varepsilon} \|y\|$$

Letting

$$x = \frac{(\lambda I - T)^{-1} y}{\|(\lambda I - T)^{-1} y\|}$$

we have $\|x\| = 1$ and

$$\|(\lambda I - T)x\| = \frac{\|y\|}{\|(\lambda I - T)^{-1} y\|} < \varepsilon$$

(b) This follows from (a) upon letting $\varepsilon \in {}^*\mathbb{R}^+$ be infinitesimal. ■

Theorem 3.6 generalizes a result given in Farrukh (1975). However, the definitions in Farrukh (1975) are incorrect. The vector in Theorem 3.6(b) is called a (unit) *ultraeigenvector* corresponding to the *ultraeigenvalue* λ. Of course, eigenvalues are special cases of ultraeigenvalues and similarly for eigenvectors.

Corollary 3.7. For any symmetric operator T on $\mathscr{H}$, we have $\sigma_p(T) \cup \sigma_c(T) \subseteq {}^*\mathbb{R}$.

Proof. Let $\lambda \in \sigma_p(T) \cup \sigma_c(T)$ and suppose the imaginary part $\operatorname{Im} \lambda \neq 0$. By Theorem 3.6(a) there exists a unit vector x such that $\|Tx - \lambda x\| < |\operatorname{Im} \lambda|$. Since T is symmetric, we have

$$\langle x, Tx \rangle = \langle Tx, x \rangle = \langle x, Tx \rangle^*$$

so $\langle x, Tx \rangle \in {}^*\mathbb{R}$. Applying Schwarz's inequality, we have

$$\begin{aligned}(\langle x, Tx \rangle - \operatorname{Re} \lambda)^2 + (\operatorname{Im} \lambda)^2 &= |\langle x, Tx \rangle - \lambda|^2 = |\langle x, (T - \lambda I)x \rangle|^2 \\ &\leq \|(T - \lambda I)x\|^2 < (\operatorname{Im} \lambda)^2\end{aligned}$$

This gives a contradiction, so $\operatorname{Im} \lambda = 0$. ■

Lemma 3.8. Let T be a symmetric operator on $\mathscr{H}$ and let x, x' be unit vectors in $\mathscr{D}(T)$. If $\lambda, \lambda' \in {}^*\mathbb{R}$ with $\lambda \neq \lambda'$, then

$$|\langle x, x' \rangle| \leq \frac{\|Tx - \lambda x\| + \|Tx' - \lambda' x'\|}{\|\lambda - \lambda'\|}$$

Proof. Let $y = Tx - \lambda x$, $y' = Tx' - \lambda' x'$. Then, since $\lambda \in {}^*\mathbb{R}$, we have

$$\begin{aligned}\lambda \langle x, x' \rangle + \langle y, x' \rangle &= \langle \lambda x + y, x' \rangle = \langle Tx, x' \rangle = \langle x, Tx' \rangle \\ &= \langle x, y' + \lambda' x' \rangle = \langle x, y' \rangle + \lambda' (x, x' \rangle\end{aligned}$$

Therefore,

$$(\lambda - \lambda') \langle x, x' \rangle = \langle x, y' \rangle - \langle y, x' \rangle$$

Hence, by Schwarz's inequality we have

$$|\lambda - \lambda'| \cdot |\langle x, x' \rangle| \leq |\langle x, y' \rangle| + |\langle y, x' \rangle| \leq \|y'\| + \|y\|$$

and the result follows. ■

Theorem 3.9. Let T be a symmetric operator on $\mathscr{H}$. (a) If x, x' are eigenvectors corresponding to distinct eigenvalues λ, λ', respectively, then $\langle x, x' \rangle = 0$. (b) If x, x' are unit ultraeigenvectors corresponding to distinct ultraeigenvalues λ, λ' and if there exists an infinite $\omega \in {}^*\mathbb{R}^+$ such that

$$|\lambda - \lambda'| \geq \omega(\|Tx - \lambda x\| + \|Tx' - \lambda' x'\|)$$

then $\langle x, x' \rangle \approx 0$.

Proof. Part (a) follows immediately from Lemma 3.8.

(b) By Corollary 3.7, $\lambda, \lambda' \in {}^*\mathbb{R}$. We then have by Lemma 3.8 that $|\langle x, x'\rangle| \leq 1/\omega \approx 0$. ∎

It is claimed in Farrukh (1975) that unit ultraeigenvectors for a self-adjoint operator corresponding to distinct ultraeigenvalues are orthogonal to within an infinitesimal. This is incorrect. Indeed, suppose $\lambda \neq \lambda'$ are ultraeigenvalues with $\lambda \approx \lambda'$. Let x be a unit ultraeigenvector corresponding to λ. Then $\lambda x \approx \lambda' x$, so $Tx \approx \lambda x \approx \lambda' x$. Hence, x is also an ultraeigenvector corresponding to λ'. But $\langle x, x\rangle = 1 \not\approx 0$. An example of such a situation is given in the next section. However, we do have the following result.

Corollary 3.10. Let T be a symmetric operator on $\mathscr{H}$. If λ, λ' are distinct ultraeigenvalues of T, then there exist unit ultraeigenvectors x, x' corresponding to λ, λ', respectively, such that $\langle x, x'\rangle \approx 0$.

Proof. Let $\omega \in {}^*\mathbb{R}^+$ be an infinite number such that $\omega \geq 2/|\lambda - \lambda'|$. Then by Theorem 3.6(a), there exist x, x' with $\|x\| = \|x'\| = 1$ such that $\|Tx - \lambda x\|, \|Tx' - \lambda' x'\| < 1/\omega^2$. Now

$$|\lambda - \lambda'| \geq \frac{2\omega}{\omega^2} > \omega(\|Tx - \lambda x\| + \|Tx' - \lambda' x'\|)$$

Hence, by Theorem 3.9(b) we have $\langle x, x'\rangle \approx 0$. ∎

4. INTERNAL INNER PRODUCT SPACES

Let $\mathscr{H}_n, n \in \mathbb{N}$, be complex Hilbert spaces and let $S = \prod_{n\in\mathbb{N}} \mathscr{H}_n$ be their Cartesian product. As in Section 2, if $(\psi_n), (\psi'_n) \in S$, we write $(\psi_n) \sim (\psi'_n)$ if $\psi_n = \psi'_n$ a.e.. We define $\Gamma(\mathscr{H}_n) \equiv \Gamma(\mathscr{H}_n : n \in \mathbb{N})$ by

$$\Gamma(\mathscr{H}_n) = S/\sim = \{[\psi_n] : (\psi_n) \in S\}$$

We define addition, scalar multiplication, and inner product on $\Gamma(\mathscr{H}_n)$ by

$$[\psi_n] + [\psi'_n] = [\psi_n + \psi'_n]$$

$$[a_n][\psi_n] = [a_n \psi_n]$$

$$\langle [\psi_n], [\psi'_n]\rangle = [\langle \psi_n, \psi'_n\rangle]$$

It is straightforward to show that $\Gamma(\mathscr{H}_n)$ is now a hyper-inner product space. We call $\Gamma(\mathscr{H}_n)$ the *internal inner product space generated by* $\{\mathscr{H}_n : n \in \mathbb{N}\}$. [Actually, $\Gamma(\mathscr{H}_n)$ is an internal subspace of ${}^*\mathscr{K}$, where $\mathscr{K} = \bigoplus_{n\in\mathbb{N}} \mathscr{H}_n$, but this introduces an unnecessary complication, since ${}^*\mathscr{K}$ is much too large for our purposes.]

Let $\Psi_n, n \in {}^*\mathbb{N}$, be an internal hypersequence in $\Gamma(\mathscr{H}_n)$. That is, there exist sequences $\psi_i^j \in \mathscr{H}_i$, $i, j \in \mathbb{N}$, such that $\Psi_n = [\psi_i^{n_i}]_i$, where $n = [n_i]$. We define the *internal sum* $\sum_{n\in *\mathbb{N}} \Psi_n$ as the internal hypersequence given by

$$\left(\sum_{n\in *\mathbb{N}} \Psi_n\right)(m) = \left[\sum_{j=0}^{m_i} \psi_i^j\right]_i$$

where $m = [m_i]$. We write $\sum_{n\in *\mathbb{N}} \Psi_n = \Psi$ if the hypersequence $m \mapsto (\sum_{n\in *\mathbb{N}} \Psi_n)(m)$ converges to Ψ. The next result shows that we can form an orthonormal "basis" for $\Gamma(\mathscr{H}_n)$ using orthonormal bases from each of the $\mathscr{H}_n$.

Theorem 4.1. Let $\mathscr{H} = \Gamma(\mathscr{H}_n)$ where $\mathscr{H}_n$ is separable, $n \in \mathbb{N}$, and let $(\psi_i^j)_j$ be an orthonormal basis for $\mathscr{H}_i$, $i \in \mathbb{N}$. Define the internal hypersequence $\Psi_n = [\psi_i^{n_i}]_i$, where $n = [n_i]$. Then Ψ_n, $n \in *\mathbb{N}$, is an orthonormal set in $\mathscr{H}$ such that for every $\Phi \in \mathscr{H}$ we have

$$\sum_{n\in *\mathbb{N}} \langle \Psi_n, \Phi \rangle \Psi_n = \Phi \tag{4.1}$$

Proof. The vectors Ψ_n are normal since

$$\|\Psi_n\|^2 = [\|\psi_i^{n_i}\|^2] = 1$$

To show orthogonality, suppose $n, m \in *\mathbb{N}$, with $n \neq m$. Since $n_i \neq m_i$ a.e., we have

$$\langle \psi_n, \psi_m \rangle = [\langle \psi_i^{n_i}, \psi_i^{m_i} \rangle] = 0$$

Hence, Ψ_n, $n \in *\mathbb{N}$, is an orthonormal set. Now the left side of (4.1) is the internal hypersequence given by

$$\left(\sum_{n\in *\mathbb{N}} (\Psi_n, \Phi\rangle \Psi_n\right)(m) = \left[\sum_{j=0}^{m_i} \langle \psi_i^j, \phi_i \rangle \psi_i^j\right]_i$$

where $\Phi = [\phi_i]$. To show convergence of (4.1), let $\varepsilon \in *\mathbb{R}^+$, where $\varepsilon = [\varepsilon_i]$, $\varepsilon_i \in \mathbb{R}^+$. Then there exists $N_i \in \mathbb{N}$ such that $m_i \geq N_i$ implies

$$\left\|\sum_{j=0}^{m_i} \langle \psi_i^j, \phi_i \rangle \psi_i^j - \phi_i\right\| < \varepsilon_i, \qquad i \in \mathbb{N}$$

Letting $N = [N_i] \in *\mathbb{N}$, if $m \geq N$, $M \in *\mathbb{N}$, we have

$$\left\|\left(\sum_{n\in *\mathbb{N}} \langle \Psi_n, \Phi \rangle \Psi_n\right)(m) - \Phi\right\| = \left[\left\|\sum_{j=0}^{m_i} \langle \psi_i^j, \phi_i) \psi_i^j - \phi_i\right\|\right] < [\varepsilon_i] = \varepsilon \quad \blacksquare$$

We call Ψ_n, $n \in *\mathbb{N}$, of Theorem 4.1 the *internal orthonormal basis generated by* (ψ_i^j), $i, j \in \mathbb{N}$. In a similar way, we can prove Parseval's equality

$$\langle \Psi, \Phi \rangle = \sum_{n\in *\mathbb{N}} \langle \Psi, \Psi_n \rangle \langle \Psi_n, \Phi \rangle$$

Let T_n be an operator on $\mathscr{H}_n$ with domain $\mathscr{D}(T_n), n \in \mathbb{N}$. Define $\mathscr{D}(T) = [\mathscr{D}(T_n)]$ and define the operator T on $\Gamma(\mathscr{H}_n)$ by $T = [T_n]$, where, of course, $[T_n][\psi_n] = [T_n \psi_n]$. We call T the *internal operator generated by* (T_n). When we say $T_n : \mathscr{H}_n \to \mathscr{H}_n$ is bounded, we mean the definition in the usual sense.

Theorem 4.2. If $T_n : \mathscr{H}_n \to \mathscr{H}_n, n \in \mathbb{N}$, are bounded, the the internal operator $T = [T_n]$ on $\mathscr{H} = \Gamma(\mathscr{H}_n)$ is bounded and normable with norm $[\|T_n\|]$. Conversely, if T is bounded, then T_n is bounded for almost all n.

Proof. For $x = [x_n] \in \mathscr{H}$ we have

$$\begin{aligned} \|Tx\| &= [\|T_n x_n\|] \leq [\|T_n\| \cdot \|x_n\|] = [\|T_n\|][\|x_n\|] \\ &= [\|T_n\|]\|x\| \end{aligned}$$

Hence, $[\|T_n\|] \in {}^*\mathbb{R}$ is a bound for T, so T is bounded. Moreover, $[\|T_n\|]$ is an upper bound for the set $\mathscr{M} = \{\|Tx\| \colon \|x\| \leq 1\}$. Let $M = [M_n] \in {}^*\mathbb{R}$ be an upper bound for $\mathscr{M}$ and suppose $M < [\|T_n\|]$. Let $\varepsilon = [\varepsilon_n] \in {}^*\mathbb{R}^+$. Then there exist $x_n \in \mathscr{H}_n$, $\|x_n\| = 1$, such that $\|T_n x_n\| > \|T_n\| - \varepsilon_n$ for almost all n. Hence, if $x = [x_n]$, we have $\|x\| = 1$ and

$$M \geq \|Tx\| > [\|T_n\|] - \varepsilon$$

Hence, $0 < [\|T_n\|] - M < \varepsilon$, which is a contradiction. Thus $[\|T_n\|] = \sup(\mathscr{M})$ and $\|T\| = [\|T_n\|]$. Conversely, suppose T is bounded and T_n is unbounded for almost all n. Then, given $M_n \in \mathbb{R}^+$, there exist $x_n \in \mathscr{H}_n$ with $\|x_n\| = 1$ such that $\|T_n x_n\| > M_n$ for almost all n. Hence, for any $M \in [M_n] \in {}^*\mathbb{R}^+$ there exists an $x = [x_n] \in \mathscr{H}$ with $\|x\| = 1$ such that $\|Tx\| > M$. This contradicts the boundedness of T, so T_n is bounded for almost all n. ■

The following theorem has a similar proof.

Theorem 4.3. An internal linear functional $f \colon \mathscr{H} \to {}^*\mathbb{C}, f = [f_n]$, is bounded if and only if $f_n : \mathscr{H}_n \to \mathbb{C}$ is bounded for almost all n.

The next result relates the resolvent set and spectrum of an internal operator with those of its generating operators. A similar theorem is presented in Farrukh (1975). Unfortunately, the definitions and proof for this result are incorrect.

Theorem 4.4. If $T = [T_n]$ is an internal operator on $\mathscr{H} = \Gamma(\mathscr{H}_n)$, then (a) $\rho(T) = [\rho(T_n)]$, (b) $\sigma_p(T) = [\sigma_p(T_n)]$, (c) $\sigma_c(T) = [\sigma_c(T_n)]$, (d) $\sigma_r(T) = [\sigma_t(T_n)]$.

Proof. (a) Suppose $\lambda \in [\rho(T_n)]$, where $\lambda = [\lambda_n]$. Then $\lambda_n I - T_n$ is a bijection of $\mathscr{D}(T_n)$ onto $\mathscr{H}_n$ for almost all n. It easily follows that $\lambda I - T$ is a bijection of $\mathscr{D}(T_n)$ onto $\mathscr{H}$, and $(\lambda I - T)^{-1} = [(\lambda_n I - T_n)^{-1}]$. Since $(\lambda_n I - T_n)^{-1}$ is bounded for almost all n, it follows from Theorem 4.2 that $(\lambda I - T)^{-1}$ is bounded. Hence, $\lambda \in \rho(T)$ and $[\rho(T_n)] \subseteq \rho(T)$. Conversely, if $\lambda \in [\lambda_n] \notin [\rho(T_n)]$, then $\lambda_n I - T_n$ is not bijective or $(\lambda_n I - T_n)^{-1}$ exists but is unbounded for almost all n. In the first case, it follows that $\lambda I - T$ is not bijective and in the second case, applying Theorem 4.2, we conclude that $(\lambda I - T)^{-1}$ is unbounded. Hence, $\lambda \notin \rho(T)$ and the result follows. The proofs of (b), (c), and (d) are similar. ■

Theorem 4.5. If $T = [T_n]$ is an internal operator on $\mathscr{H} = \Gamma(\mathscr{H}_n)$ with dense domain, then $T^* = [T_n^*]$.

Proof. Since T has a dense domain, a straightforward argument shows that $\mathscr{D}(T_n)$ is dense for almost every n. We now work on this set of measure 1. Let $x = [x_n] \in \mathscr{D}([T_n^*])$. Then there exist $y_n \in \mathscr{H}_n$ such that $\langle x_n, T_n z_n \rangle = \langle y_n, z_n \rangle$ for all $z_n \in \mathscr{D}(T_n)$. Letting $y = [y_n] \in \mathscr{H}$, we have for every $z = [z_n] \in \mathscr{D}(T)$ that

$$\begin{aligned} \langle x, Tz \rangle &= [\langle x_n, T_n z_n \rangle] = [\langle y_n, z_n \rangle] = \langle y, z \rangle \\ &= [\langle T_n^* x_n, z_n \rangle] = \langle [T_n^*] x, z \rangle \end{aligned}$$

Hence, $x \in \mathscr{D}(T)$ and $T^* x = [T_n^*] x$. Conversely, suppose $x = [x_n] \notin \mathscr{D}([T_n^*])$. Then for almost all $n \in \mathbb{N}$, $x_n \notin \mathscr{D}(T_n^*)$. Hence, by the Riesz theorem (Dunford and Schwartz, 1958; Reed and Simon, 1972), the linear functional $f_n : \mathscr{D}(T_n) \to \mathbb{C}$ given by $f_n(z_n) = \langle x_n, T_n z_n \rangle$ is unbounded. Now define the internal linear functional $f: \mathscr{D}(T) \to {}^*\mathbb{C}$ by $f = [f_n]$. Then for every $z \in \mathscr{D}(T)$

$$f(z) = [f_n(z_n)] = [\langle x_n, T_n z_n \rangle] = \langle x, Tz \rangle$$

Then by a slight modification of Theorem 4.3 [where we replace $\mathscr{H}$ by $\mathscr{D}(T)$ and $\mathscr{H}_n$ by $\mathscr{D}(T_n)$] we conclude that f is unbounded. But then $x \notin \mathscr{D}(T^*)$, since otherwise there would exist a $y \in \mathscr{H}$ such that $f(x) = \langle y, z \rangle$ for every $z \in \mathscr{D}(T)$. We would then have from Schwarz's inequality that

$$|f(z)| = |\langle y, z \rangle| \leq \|y\| \cdot \|z\|$$

which implies that f is bounded. But this is a contradiction. We conclude that $\mathscr{D}(T^*) = \mathscr{D}([T_n^*])$ and $T^* = [T_n^*]$. ■

Corollary 4.6. An internal operator $T = [T_n]$ on $\Gamma(\mathscr{H}_n)$ is self-adjoint if and only if T_n is self-adjoint a.e.

The next corollary follows from the standard theory of self-adjoint operators, Theorem 4.4, and Corollary 4.6.

Corollary 4.7. If T is an internal self-adjoint operator on $\Gamma(\mathscr{H}_n)$, then $\sigma(T) \subseteq {}^*\mathbb{R}$ and $\sigma_r(T) = \emptyset$.

It follows from Corollary 4.7 that $\sigma(T)$ consists entirely of ultraeigenvalues.

We now illustrate Corollary 3.10 with an example. Let $\mathscr{K} = L^2[0, 1]$ and let $\mathscr{H} = {}^*\mathscr{K}$. Then $\mathscr{H}$ is a special case of an internal inner product space $\Gamma(\mathscr{H}_n)$ in which $\mathscr{H}_n = \mathscr{H}$ for all $n \in \mathbb{N}$. Let $A: \mathscr{K} \to \mathscr{K}$ be the self-adjoint operator defined by $(Af)(\lambda) = \lambda f(\lambda)$. Then $T = {}^*A$ is an internal self-adjoint operator on $\mathscr{H}$. Since $\sigma(A) = \sigma_c(A) = [0, 1]$, it follows from Theorem 4.4 that

$$\sigma(T) = \sigma_c(T) = {}^*[0, 1] = \{\lambda \in {}^*\mathbb{R} \colon 0 \le \lambda \le 1\}$$

Define the vectors $f_n \in \mathscr{H}$ by

$$f_n(\lambda) = \begin{cases} \sqrt{n} & \text{for} \quad 0 \le \lambda \le 1/n \\ 0 & \text{otherwise} \end{cases}$$

and let $f = [f_n] \in \mathscr{H}$. Since $\|f_n\| = 1$, $n \in \mathbb{N}$, $\|f\| = 1$. Since $\|Af_n\| = 1/\sqrt{3}n$, we have $\|Tf\| = [1/\sqrt{3}n]$. Hence, $Tf \approx 0$, so f is a unit ultraeigenvector corresponding to the ultraeigenvalue 0. Next let $\lambda_0 = [1/n]$. Then λ_0 is an ultraeigenvalue of T and $\lambda_0 \approx 0$. Define the vectors $g_n \in \mathscr{H}$ by

$$g_n(\lambda) = \begin{cases} \sqrt{n} & \text{for} \quad 1/n \le \lambda \le 2/n \\ 0 & \text{otherwise} \end{cases}$$

and let $g = [g_n] \in \mathscr{H}$. Then $\|g\| = 1$ and since $\langle f_n, g_n \rangle = 0$, we have $\langle f, g \rangle = 0$. Moreover, since

$$\left\| Ag_n - \frac{1}{n} g_n \right\| = \frac{1}{\sqrt{3}n}$$

we have

$$\|Tg - \lambda_0 g\| = \left[\frac{1}{\sqrt{3}n} \right] \approx 0$$

Hence, $Tg \approx \lambda_0 g$, so g is a unit ultraeigenvector corresponding to the ultraeigenvalue λ_0.

Let $T = [T_n]$ be an internal self-adjoint operator on $\mathscr{H} = \Gamma(\mathscr{H}_n)$. Following quantum mechanical terminology, we call T an *observable*. An *internal state* is an internal self-adjoint operator $D = [D_n]$, where D_n is a positive trace class operator on $\mathscr{H}_n$ with trace 1. If D_n is a one-dimensional projection a.e., then D is a *pure state* and we can identify D with a unit vector $\Psi = [\psi_n]$, where $\psi_n \in \mathscr{H}_n$, $\|\psi_n\| = 1$. Since each T_n is self-adjoint, there is an associated spectral measure P_n. Then P_n is a projection-valued measure from the Borel σ-algebra $\mathscr{B}(\mathbb{R})$ to the lattice of projections $\mathscr{L}(\mathscr{H}_n)$ on $\mathscr{H}_n$. It is well known that $\mu_n(A) = \operatorname{tr}(D_n P_n(A))$, $A \in \mathscr{B}(\mathbb{R})$ is a probability measure on $\mathscr{B}(\mathbb{R})$. Now ${}^*\mathscr{B}(\mathbb{R})$ is an internal algebra of subsets of ${}^*\mathbb{R}$

consisting of sets of the form $B = [B_n]$, $B_n \in \mathscr{B}(\mathbb{R})$. Moreover, $\mu_D^T = [\mu_n]$ is an internal finitely additive probability measure on $^*\mathscr{B}(\mathbb{R})$. If D is a pure state with corresponding unit vector Ψ, we use the notation μ_Ψ^T. In this case

$$\mu_n(A) = \langle P_n(A)\psi_n, \psi_n \rangle = \|P_n(A)\psi_n\|^2$$

We now let $(^*\mathbb{R}, L_D^T(^*\mathscr{B}(\mathbb{R})), L(\mu_D^T))$ be the corresponding Loeb probability space. Then $L_D^T(^*\mathscr{B}(\mathbb{R}))$ is a σ-algebra on $^*\mathbb{R}$ and $L(\mu_D^T)$ is a real-valued probability measure on $L_D^T(^*\mathscr{B}(\mathbb{R}))$. Moreover, for every $A \in \mathscr{B}(\mathbb{R})$ we have

$$L(\mu_D^T)(^*A) = \mathrm{st}(\mu_D^T(^*A)) = \mathrm{st}[\mathrm{tr}(D_n P_n(A))]$$

or in the case of a pure state

$$L(\mu_\Psi^T)(^*A) = \mathrm{st}[\|P_n(A)\psi_n\|^2]$$

We call $L(\mu_D^T)$ the *probability distribution* of the observable T in the state D. If we define P^T on $^*\mathscr{B}(\mathbb{R})$ by $P^T([A_n]) = [P_n(A_n)]$, then P^T is an internal finitely additive projection-valued measure on $^*\mathscr{B}(\mathbb{R})$. Moreover, for every $A = [A_n] \in {}^*\mathscr{B}(\mathbb{R})$ we have

$$^\circ\mathrm{tr}(DP^T(A)) = {}^\circ[\mathrm{tr}(D_n P_n(A))] = L(\mu_D^T)(A)$$

We interpret $P^T(A)$ as the quantum event that T has a value in the set $A \in {}^*\mathscr{B}(\mathbb{R})$ and $L(\mu_D^T)(A)$ is the probability of this event in the state D. As in the standard quantum logic approach, we interpret the set of all internal projections $\mathscr{L}(\mathscr{H})$ on $\mathscr{H}$ as the set of quantum events (or propositions) for a quantum system. It is not hard to show that $\mathscr{L}(\mathscr{H})$ is an atomistic orthomodular lattice (Beltrametti and Cassinelli, 1981; Pták and Pulmannová, 1991; Morash, 1975).

Theorem 4.8. Let T be an observable on $\mathscr{H} = \Gamma(\mathscr{H}_n)$. (a) $\lambda \in \sigma_p(T)$ if and only if $P^T(\{\lambda\}) \neq 0$. (b) $\lambda \in \sigma_c(T)$ if and only if $P^T(\{\lambda\}) = 0$ and $P^T(A) \neq 0$ for every internal open set $A \in {}^*\mathscr{B}(\mathbb{R})$ containing λ.

Proof. (a) By Theorem 4.4, $\lambda \in \sigma_p(T)$ if and only if $\lambda_n \in \sigma_p(T_n)$ a.e. But by the standard theory $\lambda_n \in \sigma_p(T_n)$ if and only if $P_n(\{\lambda_n\}) \neq 0$. The last statement holds a.e. if and only if $P^T(\{\lambda\}) \neq 0$. (b) If $\lambda \in \sigma_c(T)$, then $\lambda \notin \sigma_p(T)$, so by part (a), $P^T(\{\lambda\}) = 0$, and by Theorem 4.4, $\lambda_n \in \sigma_c(T_n)$ a.e. By the standard theory, if $\lambda_n \in \sigma_c(T_n)$ and A_n is an open set in $\mathbb{R}$ containing λ, then $P_n(A_n) \neq 0$. Hence, if $A = [A_n]$ is an internal open set, then $P^T(A) \neq 0$. Conversely, suppose $\lambda \notin \sigma_c(T)$. If $\lambda \in \sigma_p(T)$, we are finished, so suppose $\lambda \notin \sigma_p(T)$. Then $\lambda \in \rho(T)$, so by Theorem 4.4, $\lambda_n \in \rho(T_n)$ a.e. By the standard theory, if $\lambda_n \in \rho(T_n)$, then there exists an open set $A_n \subseteq \mathbb{R}$ containing λ_n such that $P_n(A_n) = 0$. Therefore, $P^T(A) = [P_n(A_n)] = 0$. ■

Theorem 4.9. Let T be an observable on $\mathscr{H} = \Gamma(\mathscr{H}_n)$. (a) If $\lambda \in \sigma_p(T)$, then a unit vector $x \in \mathscr{H}$ is an eigenvector corresponding to the eigenvalue λ if and only if $L(\mu_x^T)(\{\lambda\}) = 1$. (b) If $\alpha \in \sigma_c(T)$ and x is a unit ultraeigenvector corresponding to the ultraeigenvalue α, then there exists an infinitesimal $\delta > 0$ such that $L(\mu_x^T)(\alpha - \delta, \alpha + \delta) = 1$, where

$$(\alpha - \delta, \alpha + \delta) = \{\lambda \in {}^*\mathbb{R}: \alpha - \delta < \lambda < \alpha + \delta\}$$

Proof. (a) Let $\lambda = [\lambda_n]$ and $x = [x_n]$. Then $Tx = \lambda x$ if and only if $T_n x_n = \lambda_n x_n$ a.e. By the standard theory $T_n x_n = \lambda_n x_n$ if and only if $P_n(\{\lambda_n\})x_n = x_n$. But $P_n(\{\lambda_n\})x_n = x_n$ if and only if

$$\mu_n(\{\lambda_n\}) = \|P_n(\{\lambda_n\})x_n\|^2 = \|x_n\|^2 = 1$$

(b) Suppose $\alpha \in \sigma_c(T)$ and $\|Tx - \alpha x\| < \varepsilon$, where $\|x\| = 1$ and $\varepsilon > 0$ is infinitesimal. If $\varepsilon = [\varepsilon_n]$, $x = [x_n]$, $\alpha = [\alpha_n]$, then $\|T_n x_n - \alpha_n x_n\| < \varepsilon_n$ a.e. Letting $\delta_n = \varepsilon_n^{1/2}$ and applying the spectral theorem for self-adjoint operators (Dunford and Schwartz, 1958; Reed and Simon, 1972) gives for $B_n = \{\lambda \in \mathbb{R}: |\lambda - \alpha_n| \geq \delta_n\}$

$$\begin{aligned}\varepsilon_n^2 > \|(T_n - \alpha_n I)x_n\|^2 &= \int_{-\infty}^{\infty} |\lambda - \alpha_n|^2 \|P_n(d\lambda)x_n\|^2 \\ &\geq \int_{B_n} |\lambda - \alpha_n|^2 \|P_n(d\lambda)x_n\|^2 \\ &\geq \delta_n^2 \int_{B_n} \|P_n(d\lambda)x_n\|^2 = \delta_n^2 \|P_n(B_n)x_n\|^2\end{aligned}$$

Hence, $\|P_n(B_n)x_n\| < \varepsilon_n$ and it follows that

$$\|P_n(\alpha_n - \delta_n, \alpha_n + \delta_n)x_n\|^2 \geq 1 - \varepsilon_n$$

Therefore, if $\delta = [\delta_n]$, we have

$$\begin{aligned}L(\mu_x^T)(\alpha - \delta, \alpha + \delta) &= \mathrm{st}(\|P_n(\alpha_n - \delta_n, \alpha_n + \delta_n)x_n\|^2) \\ &\geq \mathrm{st}(1 - \varepsilon) = 1\end{aligned}$$

and the result follows. ■

Theorem 4.9(b) says that if x is a unit ultraeigenvector corresponding to the ultraeigenvalue α, then in the state x, the observable T has a value infinitesimally close to α with certainty.

5. NONRELATIVISTIC NONSTANDARD FOCK SPACE

Let $\mathscr{H}$ be a Hilbert space corresponding to the one-particle states of a quantum particle. For $n \in \mathbb{N}$, we denote the nth symmetric tensor product of $\mathscr{H}$ by $\mathscr{H}^{(s)n}$ and the nth antisymmetric tensor product of $\mathscr{H}$ by $\mathscr{H}^{(a)n}$, where $\mathscr{H}^{\langle s\rangle 0} = \mathscr{H}^{(a)0} = \mathbb{C}$. We denote the symmetric and antisymmetric n or fewer particle spaces by

$$\mathscr{H}_s^n = \bigoplus_{i=0}^{n} \mathscr{H}^{(s)i}$$

$$\mathscr{H}_a^n = \bigoplus_{i=0}^{n} \mathscr{H}^{(a)i}$$

respectively. The *symmetric* and *antisymmetric nonstandard Fock spaces over* $\mathscr{H}$ are defined by

$$\Gamma_s(\mathscr{H}) = \Gamma(\mathscr{H}_s^n : n \in \mathbb{N})$$

$$\Gamma_a(\mathscr{H}) = \Gamma(\mathscr{H}_a^n : n \in \mathbb{N})$$

respectively.

For $m, n \in \mathbb{N}$, $m < n$, let P_{nm}^s, P_{nm}^a be the natural projection operators from $\mathscr{H}_s^n$ to $\mathscr{H}_s^m$ and from $\mathscr{H}_a^n$ to $\mathscr{H}_a^m$, respectively. Define $U_n^s: {}^*\mathscr{H}_s^n \to \Gamma_s(\mathscr{H})$ by $U_n^s([\psi_i]) = [\phi_i]$, where

$$\phi_i = \begin{cases} \psi_i & \text{for } i \geq n \\ 0 & \text{for } i < n \end{cases}$$

and define $U_n^a: {}^*\mathscr{H}_a^n \to \Gamma_s(\mathscr{H})$ in this same way. Let $\mathscr{F}_s = \bigoplus_{i=0}^{\infty} \mathscr{H}^{(s)i}$ and $\mathscr{F}_a = \bigoplus_{i=0}^{\infty} \mathscr{H}^{(a)i}$ be the standard symmetric and antisymmetric Fock spaces. We denote elements of $\mathscr{F}_s$ and $\mathscr{F}_a$ by (ψ_n), where $\psi_n \in \mathscr{H}^{(s)n}$, $\mathscr{H}^{(a)n}$ respectively. Define $U^s: \mathscr{F}_s \to \Gamma_s(\mathscr{H})$ by $U^s((\psi_n)) = [\phi_n]$, where $\phi_n = \bigoplus_{i=0}^{n} \psi_i$. We also define U^a: $\mathscr{F}_a \to \Gamma_a(\mathscr{H})$ in this same way. The next result shows that ${}^*\mathscr{H}_s^n$ and $\mathscr{F}_s$ can be unitarily imbedded (to within infinitesimals) into $\Gamma_s(\mathscr{H})$ and similarly for ${}^*\mathscr{H}_a^n$ and $\mathscr{F}_a$.

Theorem 5.1. (a) The range of U_m^s and U^s are given by

$$\operatorname{ran}(U_m^s) = \{[\phi_i] \in \Gamma_s(\mathscr{H}): P_{nm}^s \phi_n = \phi_n, n \geq m\}$$

$$\operatorname{ran}(U^s) = \{[\phi_i] \in \Gamma_s(\mathscr{H}): [\phi_i] \text{ is finite, } P_{nm}^s \phi_n = \phi_m, n \geq m\}$$

and similarly for $\operatorname{ran}(U_m^a)$, $\operatorname{ran}(U^a)$.

(b) $U_n^s: {}^*\mathscr{H}_s^n \to \operatorname{ran}(U_n^s)$ is unitary and U^s: $\mathscr{F}_s \to \operatorname{ran}(U^s)$ is unitary to within infinitesimals. Similar results hold for U_n^a and U^a.

Proof. (a) If $[\phi_i] \in \operatorname{ran}(U_m^s)$, then $\phi_n \in \mathscr{H}_s^m$ for all $n \geq m$, so $P_{nm}^s \phi_n = \phi_n$ for all $n \geq m$. Conversely, suppose $P_{nm}^s \phi_n = \phi_n$ for all $n \geq m$. Then $\phi_n \in \mathscr{H}_s^m$ for all $n \geq m$. Hence, $[\phi_i] \in {}^*\mathscr{H}_s^m$ and $U_m^s([\phi_i]) = [\phi_i]$. A similar result holds for $\operatorname{ran}(U_m^a)$. Now suppose $[\phi_i] \in \operatorname{ran}(U^s)$. Then $\phi_n = \bigoplus_{i=0}^{n} \psi_i$, where $(\psi_i) \in \mathscr{F}_s$. Hence,

$$\|\phi_n\|^2 = \sum_{i=0}^{n} \|\psi_i\|^2 \leq \|(\psi_i)\|^2$$

so $[\phi_i]$ is finite. Moreover, for $n \geq m$ we have

$$P^s_{nm}\phi_n = P^s_{nm} \bigoplus_{i=0}^{n} \psi_i = \bigoplus_{i=0}^{m} \psi_i = \phi_m$$

Conversely, suppose $[\phi_i] \in \Gamma_s(\mathscr{H})$, $(\phi_i]$ is finite, and $P^s_{nm}\phi_n = \phi_m, n \geq m$. Let $\psi_i = \phi_i - \phi_{i-1}$, $i \in \mathbb{N}\backslash\{0\}$, $\psi_0 = \phi_0$. Then for $i \in \mathbb{N}\backslash\{0\}$, we have

$$\psi_i = \phi_i - P_{i(i-1)}\phi_i = (I - P_{i(i-1)})\phi_i \in \mathscr{H}^{(s)i}$$

and

$$\begin{aligned}\phi_n &= \phi_0 + (\phi_1 - \phi_0) + \cdots + (\phi_{n-1} - \phi_{n-2}) + (\phi_n - \phi_{n-1}) \\ &= \bigoplus_{i=0}^{n} \psi_i\end{aligned}$$

Since $[\phi_n]$ is finite, there exists an $M \in \mathbb{R}^+$ such that $\|\phi_n\| \leq M$ a.e. Hence,

$$\sum_{i=0}^{n} \|\psi_i\|^2 = \|\phi_n\|^2 \leq M \text{ a.e.}$$

But it now easily follows that this inequality holds for all $n \in \mathbb{N}$. Thus $(\psi_n) \in \mathscr{F}_s$ and $U^s((\psi_n)) = [\phi_n]$. We conclude that $[\phi_n] \in \operatorname{ran}(U^s)$. The result for $\operatorname{ran}(U^a)$ is similar. (b) It is clear that U^s_n is unitary. To show that U^s is unitary to within infinitesimals, we have

$$\begin{aligned}\langle U^s((\psi_n)), U^s((\psi'_n))\rangle &= \left\langle \left[\bigoplus_{i=0}^{n} \psi_i\right], \left[\bigoplus_{i=0}^{n} \psi_i\right]\right\rangle \\ &= \left[\left\langle \bigoplus_{i=0}^{n} \psi_i, \bigoplus_{i=0}^{n} \psi'_i\right\rangle\right] \\ &= \left[\sum_{i=0}^{n} \langle \psi_i, \psi'_i\rangle\right] \approx \sum_{i=0}^{\infty} \langle \psi_i, \psi'_i\rangle \\ &= \langle (\psi_n), (\psi'_n)\rangle\end{aligned}$$

where the $\approx$ relation follows from Lemma 2.4 and the fact that

$$\lim_{n \to \infty} \sum_{i=0}^{n} \langle \psi_i, \psi'_i\rangle = \sum_{i=0}^{\infty} \langle \psi_i, \psi'_i\rangle \quad \blacksquare$$

Theorem 5.1 shows that the nonstandard Fock spaces contain the standard Fock spaces and moreover characterizes the subspaces of $\Gamma_s(\mathscr{H})$ and $\Gamma_a(\mathscr{H})$ corresponding to $\mathscr{F}_s$ and $\mathscr{F}_a$, respectively. Notice that it follows from Theorem 5.1(b) that

$$\operatorname{st}\langle U^s\psi, U^s\psi'\rangle = \langle \psi, \psi'\rangle$$

for all $\psi, \psi' \in \mathscr{F}_s$ and similarly for $\mathscr{F}_a$.

In nonrelativistic quantum mechanics, the one-particle space (neglecting spin) usually has the form $\mathscr{H} = L^2(\mathbb{R}^3)$. We first treat the symmetric case. For $n \in \mathbb{N}$, $n \geq 2$, $\Phi \in \mathscr{H}^{(s)n}$ if and only if $\Phi: \mathbb{R}^{3n} \to \mathbb{C}$ is measurable,

$$\int \cdots \int |\Phi(x_i, \ldots, x_n)|^2 \, dx_1 \cdots dx_n < \infty$$

and Φ is symmetric with respect to its variables $x_1, \ldots, x_n$. Notice that we use the notation $\int dx$ for $\iiint dx_1 \, dx_2 \, dx_3$. Let $n, m \in \mathbb{N}$ with $1 \leq m \leq n$. For $\Phi \in \mathscr{H}^{s(m)}$ and $f \in \mathscr{H}$, define $a_n(f)\Phi$ by

$$(a_n(f)\Phi)(x_1, \ldots, x_{m-1}) = \sqrt{m} \int dx \, f^*(x)\Phi(x, x_1, \ldots, x_{m-1})$$

and for $\Phi \in \mathscr{H}^{(s)0} = \mathbb{C}$, $a_n(f)\Phi = 0$. Also, define $a_0(f)$ on $\mathscr{H}^{(s)0} = \mathbb{C}$ by $a_0(f) = 0$. If we extend $a_n(f)$ by linearity, then we obtain a bounded operator $a_n(f): \mathscr{H}_s^n \to \mathscr{H}_s^n$. We also define $a_n^*(f): \mathscr{H}_s^n \to \mathscr{H}_s^n$ as follows. If $\Phi \in \mathscr{H}^{(s)n}$, then $a_n^*(f)\Phi = 0$. If $\Phi \in \mathscr{H}^{(s)m}$, $m < n$, we define

$$(a_n^*(f)\Phi)(x_1, \ldots, x_{m+1}) = \frac{1}{(m+1)^{1/2}} \sum_{k=1}^{m+1} f(x_k)\Phi(x_1, \ldots, \hat{x}_k, \ldots, x_{m+1})$$

where $\hat{x}_k$ means that the variable x_k is omitted. Again, extend $a_n^*(f)$ by linearity to obtain a bounded operator $a_n^*(f): \mathscr{H}_s^n \to \mathscr{H}_s^n$.

Theorem 5.2. The operator $a_n^*(f)$ is the adjoint $a_n(f)^*$ of $a_n(f)$.

Proof. Suppose $\Phi \in \mathscr{H}^{(s)m}$, $\Psi \in \mathscr{H}^{(s)p}$, where $m, p \leq n$. If $p \neq m+1$, then

$$\langle a_n^*(f)\Phi, \Psi \rangle = 0 = \langle \Phi, a_n(f)\Psi \rangle$$

Now suppose $p = m+1$, where $m < n$. Then

$$\begin{aligned}
&\langle a_n^*(f)\Phi, \Psi \rangle \\
&= \left\langle \frac{1}{(m+1)^{1/2}} \sum_{k=1}^{m+1} f(x_k)\Phi(x_1, \ldots, \hat{x}_k, \ldots, x_{m+1}), \Psi(x_1, \ldots, x_{m+1}) \right\rangle \\
&= \frac{1}{(m+1)^{1/2}} \sum_{k=1}^{m+1} \int \cdots \int dx_1 \cdots dx_{m+1} \\
&\quad \times f^*(x_k)\Phi^*(x_1, \ldots, \hat{x}_k, \ldots, x_{m+1})\Psi(x_1, \ldots, x_{m+1})
\end{aligned} \tag{5.1}$$

Moreover,

$$\begin{aligned}
&\langle \Phi, a_n(f)\Psi \rangle \\
&= \left\langle \Phi(x_1, \ldots, x_m), (m+1)^{1/2} \int dx \, f^*(x)\Psi(x, x_1, \ldots, x_m) \right\rangle \\
&= (m+1)^{1/2} \int \cdots \int dx \, dx_1 \cdots dx_m \, f^*(x)\Phi^*(x_1, \ldots, x_m)\Psi(x, x_1, \ldots, x_m)
\end{aligned} \tag{5.2}$$

The first term in the summation of (5.1) is

$$\int \cdots \int dx_1 \cdots dx_{m+1} f^*(x_1)\Phi^*(x_2, \ldots, x_{m+1})\Psi(x_1, \ldots, x_{m+1}) \quad (5.3)$$

Replacing x_1 by x and x_j by $x_{j-1}, j = 2, \ldots, m+1$, we find that (5.3) becomes

$$\int \cdots \int dx\, dx_1, \ldots, dx_m f^*(x)\Phi^*(x_1, \ldots, x_m)\Psi(x, x_1, \ldots, x_m) \quad (5.4)$$

which agrees with the integration in (5.2). Since Φ and Ψ are symmetric, each term in the summation of (5.1) has the same value as (5.4). Since there are $m + 1$ terms in the summation (5.1), we conclude that

$$\langle a_n^*(f)\Phi, \psi\rangle = \langle \Phi, a_n(f)\Psi\rangle$$

The result now follows by linearity. ■

Let $P_n: \mathscr{H}_s^n \to \mathscr{H}^{(s)n}$ be the projection of $\mathscr{H}_s^n$ onto $\mathscr{H}^{(s)n}$ and let $P_n^\perp = I - P_n$ be the projection of $\mathscr{H}_s^n$ onto $\bigoplus_{i=0}^{n-1} \mathscr{H}^{(s)i}$. It is easy to check that the commutators satisfy

$$[a_n(f), a_n(g)] = [a_n^*(f), a_n^*(g)] = 0$$

For the mixed commutators we have the following result

Theorem 5.3. The commutators $[a_n(f), a_n^*(g)]$ satisfy

$$[a_n(f), a_n^*(g)] = \langle f, g\rangle P_n^\perp - a_n^*(g)a_n(f)P_n$$

Proof. If $\Phi \in \mathscr{H}^{(s)n}$, then $a_n(f)a_n^*(g)\Phi = 0$. If $\Phi \in \mathscr{H}^{(s)m}$, $m < n$, then

$$\begin{aligned}
&(a_n(f)a_n^*(g)\Phi)(x_1, \ldots, x_m)\\
&= a_n(f)\frac{1}{(m+1)^{1/2}}\sum_{k=1}^{m+1} g(x_k)\Phi(x_1, \ldots, \hat{x}_k, \ldots, x_{m+1})\\
&= \sum_{k=1}^{m+1} \int dx_{m+1} f^*(x_{m+1})g(x_k)\Phi(x_1, \ldots, \hat{x}_k, \ldots, x_{m+1})\\
&= \sum_{k=1}^{m+1} g(x_k) \int dx_{m+1} f^*(x_{m+1})\Phi(x_1, \ldots, \hat{x}_k, \ldots, x_{m+1})\\
&\quad + \left(\int dx_{m+1} f^*(x_{m+1})g(x_{m+1})\right)\Phi(x_1, \ldots, x_m)
\end{aligned}$$

If $\Phi\in\mathscr{H}^{(s)m}$, $m\leq n$, then

$$\begin{aligned}
&(a_n^*(g)a_n(f)\Phi)(x_1,\ldots,x_m)\\
&=a_n^*(g)\sqrt{m}\int dx\, f^*(x)\Phi(x_1,\ldots,x_{m-1})\\
&=\sum_{k=1}^{m} g(x_k)\int dx\, f^*(x)\Phi(x,x_1,\ldots,\hat{x}_k,\ldots,x_m)\\
&=\sum_{k=1}^{m} g(x_k)\int dx_{m+1} f^*(x_{m+1})\Phi(x_1,\ldots,\hat{x}_k,\ldots,x_{m+1})
\end{aligned}$$

The result follows by linearity. ■

We now define the internal operators $a(f), a^*(f)$ on $\Gamma_s(\mathscr{H})$ by $a(f)=[a_n(f)]$, $a^*(f)=[a_n^*(f)]$. We call $a(f)$ and $a^*(f)$ the internal *annihilation* and *creation* operators, respectively. It follows from Theorems 4.5 and 5.2 that $a^*(f)=a(f)^*$ and hence

$$\langle a^*(f)\Phi,\Psi\rangle=\langle\Phi,a(f)\Psi\rangle$$

for all $\Phi,\Psi\in\Gamma_s(\mathscr{H})$. It is easy to show that

$$\|a_n(f)\|=\|a_n^*(f)\|=\sqrt{n}\|f\|$$

so it follows from Theorem 4.2 that $a(f)$ and $a^*(f)$ are bounded and normable with norms

$$\|a(f)\|=\|a^*(f)\|=[\sqrt{n}\|f\|]$$

Thus, unlike the standard theory, $a(f)$ and $a^*(f)$ are continuous and have domain the entire space $\Gamma_a(\mathscr{H})$. Applying Theorem 5.3, we find that the commutator of $a(f)$ and $a^*(g)$ becomes

$$[a(f),a^*(g)]=[\langle f,g\rangle P_n^\perp-a_n^*(g)a_n(f)P_n] \tag{5.5}$$

A vector $\Phi=[\Phi_n]\in\Gamma_s(\mathscr{H})$ is *large* if $\Phi_n\in\mathscr{H}^{(s)n}$ a.e., and *small* if $\Phi_n\in\bigoplus_{i=0}^{n-1}\mathscr{H}^{(s)i}$ a.e. Thus, Φ is large if only if $P_n\Phi_n=\Phi_n$ a.e., and small if and only if $P_n^\perp\Phi_n=\Phi_n$ a.e. An example of a small vector is a *finite particle vector* $\Phi=[\Phi_m]$, where there exists an $n\in\mathbb{N}$ such that $\Phi_m\in\mathscr{H}_s^n$ for almost all m. The *small subspace* of $\Gamma_s(\mathscr{H})$ is the range of the projection $P^\perp=[P_n^\perp]$ and is denoted $P^\perp\Gamma_s(\mathscr{H})$. It follows from (5.5) that on the small subspace we have

$$[a(f),a^*(g)]=\langle f,g\rangle I$$

Of course, in general we have

$$[a(f),a(g)]=[a^*(f),a^*(g)]=0$$

For $\|f\| = 1$, it is clear that the operator $N_n(f)\colon \mathscr{H}_s^n \to \mathscr{H}_s^n$ defined by $N_n(f) = a_n^*(f)a_n(f)$ is bounded and self-adjoint. Moreover, it is easy to show that $N_n(f)$ has pure point spectrum consisting of

$$\sigma(N_n(f)) = \{0, 1, \ldots, n\}$$

If we define the operator $N(f) = [N_n(f)]$ on $\Gamma_s(\mathscr{H})$, then by Theorem 4.2 and Corollary 4.6, $N(f)$ is a bounded normable internal self-adjoint operator with norm $\|N(f)\| = [n]$. Applying Theorem 4.4, we have

$$\sigma(N(f)) = \sigma_p(N(f)) = [\{0, 1, \ldots, n\}] = \{\lambda \in {}^*\mathbb{N}\colon \lambda \leq [n]\}$$

Again, unlike the standard theory, $N(f)$ is continuous and has domain all of $\Gamma_s(\mathscr{H})$.

We now consider the antisymmetric case. We again let $\mathscr{H} = L^2(\mathbb{R}^3)$. For $n \in \mathbb{N}$, $n \geq 2$, $\Phi \in \mathscr{H}^{(a)n}$ if and only if $\Phi\colon \mathbb{R}^{3n} \to \mathbb{C}$ is square integrable and Φ is antisymmetric with respect to its variables $x_1, \ldots, x_n$. For $f \in \mathscr{H}$ we define $a_n(f)\colon \mathscr{H}_a^n \to \mathscr{H}_a^n$ as in the symmetric case. We define $a_n^*(f)\colon \mathscr{H}_a^n \to \mathscr{H}_a^n$ as follows. If $\Phi \in \mathscr{H}^{(a)n}$, $a_n^*(f)\Phi = 0$. If $\Phi \in \mathscr{H}^{a(m)}$, $m < n$, then

$$\begin{aligned}
&(a_n^*(f)\Phi)(x_1, \ldots, x_{m+1}) \\
&\quad = \frac{1}{(m+1)^{1/2}} \sum_{k=1}^{m+1} (-1)^{k+1} f(x_k)\Phi(x_1, \ldots, \hat{x}_k, \ldots, x_{m+1})
\end{aligned}$$

It is easy to check that this is again an antisymmetric function. We extend $a_n^*(f)$ by linearity to obtain a bounded operator from $\mathscr{H}_a^n \to \mathscr{H}_a^n$.

Theorem 5.4. The operator $a_n^*(f)$ is the adjoint of $a_n(f)$.

Proof. As in the proof of Theorem 5.2, it suffices to check the case $\Phi \in \mathscr{H}^{(a)m}$, $\Psi \in \mathscr{H}^{(a)(m+1)}$, $m < n$. As in (5.1) and (5.2), we have

$$\begin{aligned}
&\langle a_n^*(f)\Phi, \Psi\rangle \\
&\quad = \frac{1}{(m+1)^{1/2}} \sum_{k=1}^{m+1} (-1)^{k+1} \int \cdots \int dx_1 \cdots dx_{m+1} \\
&\qquad \times f^*(x_k)\Phi^*(x_1, \ldots, \hat{x}_k, \ldots, x_{m+1})\Psi(x_1, \ldots, x_{m+1}) \qquad (5.6)
\end{aligned}$$

and

$$\begin{aligned}
\langle \Phi, a_n(f)\Psi\rangle &= (m+1)^{1/2} \int \cdots \int dx\, dx_1 \cdots dx_m \\
&\quad \times f^*(x)\Phi^*(x_1, \ldots, x_m)\Psi(x, x_1, \ldots, x_m) \qquad (5.7)
\end{aligned}$$

The first term in the summation in (5.6) is

$$\int \cdots \int dx_1 \cdots dx_{m+1} f^*(x_1)\Phi^*(x_2, \ldots, x_{m+1})\Psi(x_1, \ldots, x_{m+1}) \quad (5.8)$$

Replace x_1 by x and x_j by $x_{j-1}, j = 2, \ldots, m+1$, in (5.8) to get

$$\int \cdots \int dx\, dx_1 \cdots dx_m f^*(x)\Phi^*(x_1, \ldots, x_m)\Psi(x, x_1, \ldots, x_m) \quad (5.9)$$

The second term in the summation in (5.6) is

$$-\int \cdots \int dx_1 \cdots dx_{m+1} f^*(x_2)\Phi^*(x_1, x_3, \ldots, x_{m+1})\Psi(x_1, \ldots, x_{m+1}) \quad (5.10)$$

Replace x_2 by x and x_j by $x_{j-1}, j = 3, \ldots, m+1$, in (5.10) to get

$$-\int \cdots \int dx\, dx_1 \cdots dx_m f^*(x)\Phi^*(x_1, \ldots, x_m)\Psi(x_1, x, x_2, \ldots, x_m) \quad (5.11)$$

But since Ψ is antisymmetric, (5.11) coincides with (5.9). We thus see that each of the $m+1$ terms in (5.6) is the same as (5.9). Hence,

$$\langle a_n^*(f)\Phi, \Psi\rangle = \langle \Phi, a_n(f)\Psi\rangle$$

The result follows by linearity. ■

It is again easy to check that the anticommutators satisfy

$$[a_n(f), a_n(g)]_+ = [a_n^*(f), a_n^*(g)]_+ = 0$$

We define the projection operator $P_n: \mathscr{H}_a^n \to \mathscr{H}^{(a)n}$ as in the symmetric case and let $P_n^\perp = I - P_n$.

Theorem 5.5. The anticommutator $[a_n(f), a_n^*(g)]_+$ satisfies

$$[a_n(f), a_n^*(g)]_+ = \langle f, g\rangle P_n^\perp + a_n^*\langle g\rangle a_n(f)P_n$$

Proof. If $\Phi \in \mathscr{H}^{(a)n}$, then $a_n(f)a_n^*(g)\Phi = 0$. If $\Phi \in \mathscr{H}^{(a)m}$, $m < n$, then

$$\begin{aligned}(a_n^*(g)\Phi)&(x_1, \ldots, x_{m+1}) \\ &= \frac{1}{(m+1)^{1/2}} g(x_1)\Phi(x_2, \ldots, x_{m+1}) \\ &\quad + \frac{1}{(m+1)^{1/2}} \sum_{k=2}^{m+1} (-1)^{k+1} g(x_k)\Phi(x_1, \ldots, \hat{x}_k, \ldots, x_{m+1})\end{aligned}$$

Hence,

$$(a_n^*(g)\Phi)(x, x_1, \ldots, x_m)$$
$$= \frac{1}{(m+1)^{1/2}} g(x)\Phi(x_1, \ldots, x_m)$$
$$+ \frac{1}{(m+1)^{1/2}} \sum_{k=1}^{m} (-1)^k g(x_k)\Phi(x, x_1, \ldots, \hat{x}_k, \ldots, x_m)$$

and

$$(a_n(f)a_n^*(g)\Phi)(x_1, \ldots, x_m)$$
$$= \langle f, g\rangle \Phi(x_1, \ldots, x_m) + \sum_{k=1}^{m} (-1)^k g(x_k)$$
$$\times \int dx\, f^*(x)\Phi(x, x_1, \ldots, \hat{x}_k, \ldots, x_m)$$

If $\Phi \in \mathscr{H}^{(a)m}$, $m \leq n$, then

$$(a_n^*(g)a_n(f)\Phi)(x_1, \ldots, x_m)$$
$$= a_n^*(g)\sqrt{m} \int dx\, f^*(x)\Phi(x, x_1, \ldots, x_m)$$
$$= \sum_{k=1}^{m} (-1)^{k+1} g(x_k) \int dx\, f^*(x)\Phi(x, x_1, \ldots, \hat{x}_k, \ldots, x_m)$$

The result now follows by linearity. ■

As in the symmetric case, we define the annihilation and creation operators $a(f) = [a_n(f)]$, $a^*(f) = [a_n^*(f)]$ on $\Gamma_a(\mathscr{H})$. These satisfy the anticommutation relations

$$[a(f), a(g)]_+ = [a^*(f), a^*(g)]_+ = 0$$
$$[a(f), a^*(g)]_+ = [\langle f, g\rangle P_n^\perp + a_n^*(g)a(f)P_n]$$

It follows from (5.12) that on the small subspace $P^\perp \Gamma_a(\mathscr{H})$

$$[a(f), a^*(g)]_+ = \langle f, g\rangle I$$

For $\|f\| = 1$, the n-particle number operator $N_n(f)\colon \mathscr{H}_a^n \to \mathscr{H}_a^n$ is a bounded self-adjoint operator with spectrum

$$\sigma(N_n(f)) = \sigma_p(N_n(f)) = \{0, 1\}$$

These and the number operator $N(f) = [N_n(f)]$ are projections.

6. NONRELATIVISTIC FIELD OPERATORS

As in Section 5, we let $\mathscr{H} = L^2(\mathbb{R}^3)$. We first treat the symmetric case. Let (f_i), $i \in \mathbb{N}$, be an orthonormal basis for $\mathscr{H}$ where each f_i is continuous. For each $x \in \mathbb{R}^3$ define the bounded operators $\psi_n(x)\colon \mathscr{H}_s^n \to \mathscr{H}_s^n$, $\psi_n^*(x)\colon \mathscr{H}_s^n \to \mathscr{H}_s^n$ by

$$\psi_n(x) = \sum_{j=0}^{n} f_j(x) a_n(f_j)$$

$$\psi_n^*(x) = \sum_{j=0}^{n} f_j^*(x) a_n^*(f_j)$$

We also define the bounded self-adjoint operator $N_n\colon \mathscr{H}_s^n \to \mathscr{H}_s^n$ by

$$N_n = \sum_{j=0}^{n} a_n^*(f_j) a_n(f_j)$$

Of course, these operators depend on the chosen orthonormal basis and the order in which the basis elements are given. The next result gives explicit expressions for these operators and the proof is straightforward.

Lemma 6.1. (a) If $\Phi \in \mathscr{H}^{s(m)}$, $1 \le m \le n$, then

$$(\psi_n(x)\Phi)(x_1, \ldots, x_{m-1}) = \sqrt{m} \sum_{j=0}^{n} f_j(x) \langle f_j, \Phi(\,\cdot\,, x_1, \ldots, x_{m-1})\rangle$$

(b) If $\Phi \in \mathscr{H}^{(s)m}$, $0 \le m < n$, then

$$(\psi_n^*(x)\Phi)(x_1, \ldots, x_{m+1}) = \frac{1}{(m+1)^{1/2}} \sum_{k=1}^{m+1} \sum_{j=0}^{n} f_j^*(x) f_j(x_k) \Phi(x_1, \ldots, \hat{x}_k, \ldots, x_{m+1})$$

(c) If $\Phi \in \mathscr{H}^{(s)m}$, $1 \le m \le n$, then

$$(N_n\Phi)(x_1, \ldots, x_m) = \sum_{k=1}^{m} \sum_{j=0}^{n} f_j(x_k) \langle f_j, \Phi(\,\cdot\,, x_1, \ldots, \hat{x}_k, \ldots, x_m)\rangle$$

It is clear that

$$[\psi_n(x), \psi(y)] = [\psi_n^*(x), \psi_n^*(y)] = 0$$

The next result gives other commutation relations.

Lemma 6.2. The following commutation relations hold:

(a) $[\psi_n(x), \psi_n^*(y)] = \sum_{j=0}^{n} f_j(x) f_j^*(y) P_n^{\perp} - \psi_n^*(y)\psi_n(x) P_n$.
(b) $[N_n, \psi_n^*(x)] = \psi_n^*(x)$.
(c) $[N_n, \psi_n(x)] = -\psi_n(x)$.

Proof. (a) Applying Theorem 5.3 gives

$$[\psi_n(x), \psi_n^*(y)] = \sum_{i,j=0}^{n} f_i(x)f_j^*(y)[a_n(f_i), a_n^*(f_j)]$$

$$= \sum_{i,j=0}^{n} f_i(x)f_j^*(y)\langle f_i, f_j\rangle P_n^{\perp} - \sum_{i,j=0}^{n} f_i(x)f_j^*(y)a_n^*(f_j)a_n(f_i)P_n$$

$$= \sum_{j=0}^{n} f_j(x)f_j^*(y)P_n^{\perp} - \psi_n^*(y)\psi_n(x)P_n$$

(b) From the definitions of N_n and $\psi^*(x)$, we have

$$[N_n, \psi_n^*(x)] = \left[\sum_{i=0}^{n} a_n^*(f_i)a_n(f_i), \sum_{j=0}^{n} f_j^*(x)a_n^*(f_j)\right]$$

$$= \sum_{i,j=0}^{n} f_j^*(x)[a_n^*(f_i)a_n(f_i), a_n^*(f_j)]$$

$$= \sum_{i,j=0}^{n} f_j^*(x)a_n^*(f_i)[a_n(f_i)a_n^*(f_j)]$$

$$= \sum_{i,j=0}^{n} f_j^*(x)a_n^*(f_i)\langle f_i, f_j\rangle P_n^{\perp} + \sum_{i,j=0}^{n} f_j^*(x)a_n^*(f_i)a_n^*(f_j)a_n(f_i)P_n$$

$$= \sum_{j=0}^{n} f_j^*(x)a_n^*(f_j) = \psi_n^*(x)$$

(*c*) This follows by taking the adjoint of (b) and using the fact that N_n is self-adjoint. ■

We now define the following internal operators on $\Gamma_s(\mathscr{H})$: $\psi(x) = [\psi_n(x)]$, $\psi^*(x) = [\psi_n^*(x)]$, $N = [N_n]$. Unlike the standard theory, these operators are bounded and defined on all of $\Gamma_s(\mathscr{H})$. Notice that N is an internal self-adjoint operator. We call $\psi(x)$ and $\psi^*(x)$ *field operators.* In the definition of $\psi(x)$ we assumed that $x\in\mathbb{R}^3$. However, $\psi_n(x)$ can be thought of as an operator-valued function, so $\psi = [\psi_n]$ is an internal operator-valued function which defines $\psi(x)$ for all $x\in{}^*\mathbb{R}^3$. The same observation applies for $\psi^*(x)$,

We next define the concept of a delta function in the present framework. A *delta function* $\delta_y(x)$ is an internal function δ: ${}^*\mathbb{R}^3 \times {}^*\mathbb{R}^3 \to {}^*\mathbb{C}$ that satisfies:

(1) $\delta_x(y) = \delta_y^*(x)$ for all $x, y\in{}^*\mathbb{R}^3$.
(2) For all $y\in{}^*\mathbb{R}^3$, $\delta_y \equiv \delta_y(\cdot)\in{}^*\mathscr{H}$.
(3) For any $\phi\in\mathscr{H}$, the function $y\mapsto\langle\delta_y, \phi\rangle$ is in ${}^*\mathscr{H}$ and $\langle\delta_y, \phi\rangle \approx \phi$.

Of course in (3), $\approx$ is in the norm sense; that is, $\|\langle\delta_y, \phi\rangle - \phi\|$ is infinitesimal. Although a delta function is not unique, two delta functions are close in the following sense. If δ and δ' are delta functions, then by (3) we have for every $\phi \in \mathscr{H}$ that $\langle\delta_y, \phi\rangle \approx \langle\delta'_y, \phi\rangle$. An important example of a delta function is given by the next result.

Theorem 6.3. If (f_i) is an orthonormal basis of continuous functions for $\mathscr{H}$, then

$$\delta_y(x) = \left[\sum_{i=0}^{n} f_i(x) f_i^*(y)\right]$$

is a delta function.

Proof. It is clear that δ: $^*\mathbb{R}^3 \times {}^*\mathbb{R}^3 \rightarrow {}^*\mathbb{C}$ is internal and $\delta_x(y) = \delta_y^*(x)$. To verify (2), we observe that

$$\delta_y = \left[\sum_{i=0}^{n} f_i^*(y) f_i\right] \in {}^*\mathscr{H}$$

To verify (3), let $\phi \in \mathscr{H}$. Then

$$\langle\delta_y, \phi\rangle = \left[\sum_{i=0}^{n} \langle f_i, \phi\rangle f_i(y)\right] \in {}^*\mathscr{H}$$

Since (f_i) is an orthonormal basis, we conclude that

$$\lim_{n\rightarrow\infty} \sum_{i=0}^{n} \langle f_i, \phi\rangle f_i = \phi$$

in the norm topology. Hence, $\langle\delta_y, \phi\rangle \approx \phi$. ■

The next result shows that, in a certain sense, $\psi(x)$, $\psi^*(x)$, and N are independent of the basis (f_i). If $\Phi \in \mathscr{H}^{(s)m}$, then we can view Φ as an element of $\Gamma_s(\mathscr{H})$ in accordance with Theorem 5.1.

Theorem 6.4. If $\Phi \in \mathscr{H}^{s(m)}$, then:

(a) $(\psi(\,\cdot\,)\Phi)(x_1, \ldots, x_{m-1}) \approx \sqrt{m}\,\Phi(\,\cdot\,, x_1, \ldots, x_{m-1})$.

(b) $(\psi^*(\,\cdot\,)\Phi)(x_1, \ldots, x_{m+1})$
$$\approx \frac{1}{(m+1)^{1/2}} \sum_{k=1}^{m+1} \delta_{x_k}^*(\,\cdot\,)\Phi(x_1, \ldots, \hat{x}_k, \ldots, x_{m+1}).$$

(c) $N\Phi \approx m\Phi$.

Proof. (a) In the norm topology,

$$\lim_{n\rightarrow\infty} \sum_{j=0}^{n} \langle f_j, \Phi(\,\cdot\,, x_1, \ldots, x_{m-1})\rangle f_j = \Phi(\,\cdot\,, x_1, \ldots, x_{m-1})$$

and the result follows from Lemma 6.1(a).

(b) This follows from Lemma 6.1(b) and Theorem 6.3.

(c) Applying Lemma 6.1(c) and the fact that Φ is symmetric, the result follows. ■

It is clear that

$$[\psi(x), \psi(y)] = [\psi^*(x), \psi^*(y)] = 0$$

The next result gives other commutation relations.

Theorem 6.5. (a) $[\psi(x), \psi^*(y)] = \delta_y(x)P^{\perp} - \psi^*(y)\psi(x)P$ and on the small space $P^{\perp}\Gamma_s(\mathscr{H})$, $[\psi(x), \psi^*(y)] = \delta_y(x)I$.

(b) $[N, \psi^*(x)] = \psi^*(x)$.

(c) $[N, \psi(x)] = -\psi(x)$.

Proof. Follows from Lemma 6.2 and Theorem 6.3. ■

If $f(x)$ is integrable on $\mathbb{R}^3$ and B is an operator on $\mathscr{H}_s^n$, we define the integral of the operator-valued function $A(x) = f(x)B$ by

$$\int A(x)\, dx = \left(\int f(x)\, dx\right)B$$

Moreover, we extend this definition by linearity to finite sums $\sum f_i(x)B_i$. Since

$$\psi_n^*(x)\psi_n(x) = \sum_{i,j=0}^{n} f_i(x)f_j^*(x)a_n^*(f_j)a_n(f_i)$$

we have

$$\int \psi_n^*(x)\psi_n(x)\, dx = \sum_{i,j=0}^{n} \delta_{ij}a_n^*(f_j)a_n(f_i)$$

$$= \sum_{j=0}^{n} a_n^*(f_j)a_n(f_j) = N_n$$

It follows that

$$\int \psi^*(x)\psi(x)\, dx = \left[\int \psi_n^*(x)\psi_n(x)\, dx\right] = [N_n] = N$$

Hence, $\psi^*(x)\psi(x)$ can be interpreted as the particle density operator.

We call $\Phi_0 = [\bar{1}] \in \Gamma_s(\mathscr{H})$ the *vacuum vector*. Since by Theorem 6.5(b)

$$N\psi^*(x)\Phi_0 = [N, \psi^*(x)]\Phi_0 = \psi^*(x)\Phi_0$$

we see that $\psi^*(x)\Phi_0$ is an eigenvector of N with eigenvalue 1. Moreover, by Lemma 6.1(b) we have

$$(\psi_n^*(x)\Phi_0)(y) = \sum_{j=0}^{n} f_j^*(x)f_j(y)$$

Hence, $(\psi^*(x)\Phi_0)(y) = \delta_x(y)$ and we may interpret $\psi^*(x)$ as the creation operator that creates a particle localized at x. Similarly, $\psi(x)$ destroys a particle localized at x.

We can extend the definition of $a(f)$ to include $f \in {}^*\mathscr{H}$ in the following way. If $f = [f_n] \in {}^*\mathscr{H}$, we define $a(f) = [a_n(f_n)]$. In a similar way, we define $a^*(f) = [a_n^*(f_n)]$. If A and B are operators on $\Gamma_s(\mathscr{H})$, we write $A \approx B$ if $A\Phi \approx B\Phi$ for every $\Phi \in \mathscr{H}^{(s)n}$ and all $n \in \mathbb{N}$. If $f \approx g$, it is straightforward to show that $a(f) \approx a(g)$ and $a^*(f) \approx a^*(g)$. The next result shows that $\psi(x)$ and $\psi^*(x)$ are "densities" for $a(f)$ and $a^*(f)$, respectively.

Theorem 6.6. (a) For any $f \in \mathscr{H}$, $a(f) \approx \int f^*(x)\psi(x)\,dx$ and $a^*f \approx \int f(x)\psi^*(x)\,dx$.

(b) $a(\delta_x) = \psi(x)$, $a^*(\delta_x) = \psi^*(x)$.

Proof. (a) Since $f \mapsto a(f)$ is conjugate linear, we have

$$\int f^*(x)\psi_n(x)\,dx = \sum_{j=0}^{n} \langle f, f_j\rangle a_n(f_j) = a_n\left(\sum_{j=0}^{n} \langle f_j, f\rangle f_j\right)$$

Letting $g_n = \sum_{j=0}^{n} \langle f_j, f\rangle f_j$, since $g = [g_n] \approx f$, we have

$$\begin{aligned}\int f^*(x)\psi(x)\,dx &= \left[\int f^*(x)\psi_n(x)\,dx\right] = [a_n(g_n)]\\ &= a(g) \approx a(f)\end{aligned}$$

The result for $a^*(f)$ is similar.

(b) From the definition of δ_x, we have

$$\begin{aligned}a(\delta_x) &= \left[a_n\left(\sum_{j=0}^{n} f_j^*(x) f_j\right)\right] = \left[\sum_{j=0}^{n} f_j(x) a_n(f_j)\right]\\ &= [\psi_n(x)] = \psi(x)\end{aligned}$$

The result for $a^*(\delta_x)$ is similar. ■

The antisymmetric case is quite similar to the symmetric case. The operators $\psi_n(x)$, $\psi_n^*(x)$, $\psi(x)$, $\psi^*(x)$, N_n, and N are defined in the same way on $\mathscr{H}_a^n$, $\Gamma_a(\mathscr{H})$ as before. Essentially all the previous results of this section hold with commutators replaced by anticommutators and with an additional factor of $(-1)^{k+1}$ in Lemma 6.1(b) and Theorem 6.4(b).

7. SECOND QUANTIZATION OF OPERATORS

We again let $\mathscr{H} = L^2(\mathbb{R}^3)$ and treat the symmetric case, the results in the antisymmetric case being similar. Let A be a self-adjoint operator on $\mathscr{H}$ with domain $\mathscr{D}(A)$, and let $\mathscr{D}(A)^{(s)n} \subseteq \mathscr{H}^{(s)n}$ be the subspace generated by product vectors whose components are in $\mathscr{D}(A)$. We define $\hat{\Omega}_n(A)$ on

$\mathscr{D}(A)^{(s)n}$, $n \geq 1$, by

$$\hat{\Omega}_n(A) = A \otimes I \otimes \cdots \otimes I + I \otimes A \otimes I \otimes \cdots \otimes I + I \otimes \cdots \otimes I \otimes A$$

and $\hat{\Omega}_0(A) = 0$. Then $\hat{\Omega}(A)$ has a unique self-adjoint extension, Cook (1953), which we also denote by $\hat{\Omega}_n(A)$. We sometimes write

$$\hat{\Omega}_n(A) = A(x_1) + \cdots + A(x_n)$$

so that

$$(\hat{\Omega}_n(A)\Phi)(x_1, \ldots, x_n) = A(x_1)\Phi(x_1, \ldots, x_n) + \cdots + A(x_n)\Phi(x_1, \ldots, x_n)$$

We next define $\Omega_n(A)$ on $\mathscr{H}_s^n$ by

$$\Omega_n(A) = \hat{\Omega}_0(A) \oplus \hat{\Omega}_1(A) \oplus \cdots \oplus \hat{\Omega}_n(A)$$

and the internal self-adjoint operator $\Omega(A)$ on $\Gamma_s(\mathscr{H})$ by $\Omega(A) = [\Omega_n(A)]$.

Theorem 7.1. (a) On the intersection of their domains

$$[\Omega(A), \Omega(B)] = \Omega([A, B])$$

(b) If $f \in \mathscr{D}(A)$, then on the intersection of their domains

$$[\Omega(A), a(f)] = -a(Af)$$

$$[\Omega(A), a^*(f)] = a^*(Af)$$

Proof. (a) This is a straightforward verification.

(b) For $f \in \mathscr{D}(A)$ and $\Phi \in \mathscr{H}^{(s)m} \cap \mathscr{D}(\Omega_n(A))$, $1 \leq m \leq n$, such that $a_n(f)\Phi \in \mathscr{D}(\Omega_n(A))$ we have

$$\begin{aligned}
&([\Omega_n(A), a_n(f)]\Phi)(x_1, \ldots, x_{m-1}) \\
&= \sqrt{m}\left(\sum_{k=1}^{m-1} A(x_k)\langle f, \Phi(\cdot, x_1, \ldots, x_{m-1})\rangle\right) \\
&\quad - \left(a_n(f) \sum_{k=1}^{m} A(x_k)\Phi\right)(x_1, \ldots, x_{m-1}) \\
&= \sqrt{m}\left(\sum_{k=1}^{m-1} A(x_k)\langle f, \Phi(\cdot, x_1, \ldots, x_{m-1})\rangle\right) \\
&\quad - \sqrt{m}\Bigg(\langle f, A(\cdot)\Phi(\cdot, x_1, \ldots, x_{m-1})\rangle \\
&\quad + \sum_{k=1}^{m-1} \langle f, A(x_k)\Phi(\cdot, x_1, \ldots, x_{m-1})\rangle\Bigg) \\
&= -\sqrt{m}\,\langle f, A(\cdot)\Phi(\cdot, x_1, \ldots, x_{m-1})\rangle \\
&= -\sqrt{m}\,\langle Af, \Phi(\cdot, x_1, \ldots, x_{m-1})\rangle \\
&= -(a_n(Af)\Phi)(x_1, \ldots, x_{m-1})
\end{aligned}$$

Hence,

$$[\Omega_n(A), a_n(f)] = -a_n(Af)$$

and the result follows. The second result follows by taking adjoints. ■

If A is a self-adjoint operator on $\mathscr{H}$ and the elements of the orthonormal basis (f_i) are in $\mathscr{D}(A)$, we define the operator $(A\psi_n)(x)$ on $\mathscr{H}_s^n$ by

$$(A\psi_n)(x) \equiv A(x)\psi_n(x) = \sum_{j=0}^{n} f_j(x)a_n(Af_j)$$

Moreover, we define the internal operator $(A\psi)(x)$ on $\Gamma_s(\mathscr{H})$ by

$$(A\psi)(x) \equiv A(x)\psi(x) = [(A\psi_n)(x)]$$

As before, we can extend this definition to define $(A\psi)(x)$ for $x \in {}^*\mathbb{R}^3$.

Corollary 7.2. On the domain of $[\psi(x), \Omega(A)]$ we have $[\psi(x), \Omega(A)] = (A\psi)(x)$.

Proof. From Theorem 7.1(b) we have

$$\begin{aligned}[\psi_n(x), \Omega_n(A)] &= \sum_{j=0}^{n} f_j(x)[a_n(f_j), \Omega_n(A)] \\ &= \sum_{j=0}^{n} f_j(x)a_n(Af_j) = (A\psi_n)(x)\end{aligned}$$

The result now follows. ■

Theorem 7.3. On the domain of $\Omega(A)$ we have $\Omega(A) \approx \int \psi^*(x)A(x)\psi(x)\,dx$.

Proof. By definition

$$\int \psi^*(x)A(x)\psi(x)\,dx = \left[\int \psi_n^*(x)(A\psi_n)(x)\,dx\right]$$

Since

$$\psi^*(x)(A\psi_n)(x) = \sum_{i,j=0}^{n} f_i^*(x)f_j(x)a_n^*(f_i)a_n(Af_j)$$

we have

$$\begin{aligned}\int \psi_n^*(x)(A\psi_n)(x)\,dx &= \sum_{i,j=0}^{n} \delta_{ij}a_n^*(f_i)a_n(Af_j) \\ &= \sum_{j=0}^{n} a_n^*(f_j)a_n(Af_j)\end{aligned}$$

If $\Phi \in \mathscr{H}^{(s)m}$, $1 \le m \le n$, is in the domain of $\Omega(A)$, we have

$$\left(\int \psi_n^*(x)(A\psi_n)(x)\,dx\ \Phi\right)(x_1, \ldots, x_m)$$

$$= \left(\sum_{j=0}^{n} a_n^*(f_j)a_n(Af_j)\Phi\right)(x_1, \ldots, x_m)$$

$$= \sum_{j=0}^{n} \sum_{k=1}^{m} f_j(x_k)\langle Af_j, \Phi(\,\cdot\,, x_1, \ldots, \hat{x}_k, \ldots, x_m)\rangle$$

$$= \sum_{k=1}^{m} \sum_{j=0}^{n} f_j(x_k)\langle f_j, A\Phi(\,\cdot\,, x_1, \ldots, \hat{x}_k, \ldots, x_m)\rangle$$

As $n \to \infty$, this last expression converges in norm to

$$\sum_{k=1}^{m} (A\Phi)(\,\cdot\,, x_1, \ldots, \hat{x}_k, \ldots, x_m) = \sum_{k=1}^{m} A(x_k)\Phi(x_1, \ldots, x_m)$$

$$= (\Omega(A)\Phi)(x_1, \ldots, x_m) \quad \blacksquare$$

Corollary 7.4. On the domain of $\Omega(A)$ we have

$$\Omega(A) \approx \left[\sum_{j=0}^{n} a_n^*(f_j)a_n(Af_j)\right]$$

It follows from Corollary 7.4 that

$$\Omega(I) \approx \left[\sum_{j=0}^{n} a_n^*(f_j)a_n(f_j)\right] = N$$

The next result gives an alternative expression for $(A\psi)(x)$.

Lemma 7.5. $(A\psi)(x) \approx [\sum_{j=0}^{n} (Af_j)(x)a_n(f_j)]$.

Proof. For $\Phi \in \mathscr{H}^{(s)m}$, $1 \le m \le n$, we have

$$((A\psi_n)(x)\Phi)(x_1, \ldots, x_{m-1}) = \sum_{j=0}^{n} f_j(x)\langle Af_j, \Phi(\,\cdot\,, x_1, \ldots, x_{m-1})\rangle$$

$$= \sum_{j=0}^{n} f_j(x)\langle f_j, A\Phi(\,\cdot\,, x_1, \ldots, x_{m-1})\rangle$$

This last expression converges in norm to $A(x)\Phi(x, x_1, \ldots, x_{m-1})$.

Moreover, we have

$$\left(\sum_{j=0}^{n} (Af_j)(x)a_n(f_j)\Phi\right)(x_1, \ldots, x_m)$$

$$= \sum_{j=0}^{n} (Af_j)(x)\langle f_j, \Phi(\,\cdot\,, x_1, \ldots, x_{m-1})\rangle$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{\infty} \langle f_k, Af_j\rangle\langle f_j, \Phi(\,\cdot\,, x_1, \ldots, x_{m-1})\rangle f_k(x)$$

$$= \sum_{k=0}^{\infty} f_k(x) \sum_{j=0}^{n} \langle Af_k, f_j\rangle\langle f_j, \Phi(\,\cdot\,, x_1, \ldots, x_{m-1})\rangle$$

By Parseval's equality, this last expression converges in norm to

$$\sum_{k=0}^{\infty} \langle Af_k, \Phi(\,\cdot\,, x_1, \ldots, x_{m-1})\rangle f_k(x)$$

$$= \sum_{k=0}^{\infty} \langle f_k, A\Phi(\,\cdot\,, x_1, \ldots, x_{m-1})\rangle f_k(x)$$

$$= A(x)\Phi(x, x_1, \ldots, x_{m-1}) \quad \blacksquare$$

Because of Lemma 7.5 we could have defined

$$(A\psi_n)(x) = \sum_{j=0}^{n} (Af_j)(x)a_n(f_j)$$

and Theorem 7.3 still holds.

Theorem 7.6. If H and A are self-adjoint operators on $\mathscr{H}$, then for every $t \in \mathbb{R}$ we have:

(a) $\Omega(e^{itH}Ae^{-itH}) = e^{it\Omega(H)}\Omega(A)e^{-it\Omega(H)}$.
(b) $a(e^{-itH}f) = e^{-it\Omega(H)}a(f)e^{it\Omega(H)}$.
(c) $a^*(e^{-itH}f) = e^{-it\Omega(H)}a^*(f)e^{it\Omega(H)}$.

Proof. (a) Applying the definitions, we have

$$\hat{\Omega}_n(e^{itH}Ae^{-itH}) = e^{itH}Ae^{-itH} \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes e^{itH}Ae^{-itH}$$

Moreover,

$$\exp[it\hat{\Omega}_n(H)] = \exp[it(H \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes H)]$$

$$= \exp[(itH) \otimes I \otimes \cdots \otimes I] \cdots \exp[I \otimes \cdots \otimes I \otimes (itH)]$$

$$= \exp(itH) \otimes \cdots \otimes \exp(itH)$$

Hence,

$$e^{it\hat{\Omega}_n(H)}\hat{\Omega}_n(A)e^{-it\hat{\Omega}_n(H)} = e^{itH}Ae^{-itH}\otimes I\otimes\cdots\otimes I + I\otimes\cdots\otimes I\otimes e^{itH}Ae^{-itH}$$
$$= \hat{\Omega}_n(e^{itH}Ae^{-itH})$$

By linearity we obtain

$$e^{it\Omega_n}(H)\Omega_n(A)e^{-it\Omega_n(H)} = \Omega_n(e^{itH}Ae^{-itH})$$

and the result follows.

(b) If $\Phi\in\mathscr{H}^{(s)m}$, $1\le m\le n$, then since H is self-adjoint, we have

$$(e^{-it\Omega_n(H)}a_n(f)e^{it\Omega_n(H)}\Phi)(x_1,\ldots,x_{m-1})$$
$$= (e^{-itH(x_1)}\cdots e^{-itH(x_{m-1})}a_n(f)e^{itH(x_1)}\cdots e^{itH(x_m)}\Phi)(x_1,\ldots,x_{m-1})$$
$$= e^{-itH(x_1)}\cdots e^{-itH(x_{m-1})}\sqrt{m}$$
$$\times\int f^*(x)e^{itH(x)}e^{itH(x_1)}\cdots e^{itH(x_{m-1})}\Phi(x,x_1,\ldots,x_{m-1})\,dx$$
$$= \sqrt{m}\int (e^{-itH}f)^*(x)\Phi(x,x_1,\ldots,x_{m-1})\,dx$$
$$= (a_n(e^{-itH}f)\Phi)(x_1,\ldots,x_{m-1})$$

Hence,

$$a_n(e^{-itH})f) = e^{-it\Omega_n(H)}a_n(f)e^{it\Omega_n(H)}$$

and the result follows.

(c) Take the adjoint of (b). ■

We can extend these results to two- or higher-particle interactions. For example, let $A(x_1,x_2)$ be a self-adjoint operator on $\mathscr{H}^{(s)2}$, where $A(x_1,x_2)=A(x_2,x_1)$; that is, $(A\phi)(x_1,x_2)=(A\phi)(x_2,x_1)$ for all $\phi\in\mathscr{H}^{(s)2}$. Similar to our previous method, we define $\hat{\Omega}_0(A)=\hat{\Omega}_1(A)=0$ and on $\mathscr{H}^{(s)n}$, $n\ge 2$,

$$\hat{\Omega}(A) = \sum_{\substack{i,j=1\\ i<j}}^{n} A(x_i,x_j) = \frac{1}{2}\sum_{\substack{i,j=1\\ i\ne j}}^{n} A(x_i,x_j)$$

We then define $\Omega_n(A)$ on $\mathscr{H}_s^n$ by

$$\Omega_n(A) = \hat{\Omega}_0(A)\otimes\cdots\otimes\hat{\Omega}_n(A)$$

and the internal operator $\Omega(A)$ on $\Gamma_s(\mathscr{H})$ is defined as $\Omega(A)=[\Omega_n(A)]$. Since

$$\psi_n(x)\psi_n(x') = \sum_{i,j=0}^{n} f_i(x)f_j(x')a_n(f_i)a_n(f_j)$$

as in the one-particle case (see remark following Lemma 7.5), we define

$$A(x, x')\psi_n(x)\psi_n(x') = \sum_{i,j=0}^{n} (Af_i f_j)(x, x')a_n(f_i)a_n(f_j)$$

The proof of the following result is similar to that of Theorem 7.3.

Theorem 7.7. On the domain of $\Omega(A)$ we have

$$\Omega(A) \approx \frac{1}{2} \iint \psi^*(x')\psi^*(x)A(x, x')\psi(x)\psi(x')\, dx\, dx'$$

We next briefly discuss dynamics. Let H be the one-particle Hamiltonian on $\mathscr{H}$. For $t \in \mathbb{R}$, we define

$$\psi_n(x, t) = e^{it\Omega_n(H)/\hbar}\psi_n(x)e^{-it\Omega_n(H)/\hbar}$$

and the time-dependent field operator as the internal operator $\psi(x, t) = [\psi_n(x, t)]$. It follows from Theorem 7.6(b) that

$$\psi_n(x, t) = \sum_{j=0}^{n} f_j(x)a_n(e^{itH/\hbar}f_j)$$

The equation of motion is given by

$$\frac{\partial}{\partial t}\psi_n(x, t) = e^{it\Omega_n(H)/\hbar}\psi_n(x)\left(\frac{-i\Omega_n(H)}{\hbar}\right)e^{-it\Omega_n(H)/\hbar}$$

$$+\frac{i\Omega_n(H)}{\hbar}e^{it\Omega_n(H)/\hbar}\psi_n(x)e^{-it\Omega_n(H)/\hbar}$$

$$= -\frac{i}{\hbar}[\psi_n(x, t), \Omega_n(H)]$$

Hence

$$i\hbar\frac{\partial}{\partial t}\psi(x, t) = [\psi(x, t), \Omega(H)]$$

We also have the equal-time commutation relations

$$[(\psi(x, t), \psi(y, t)] = [\psi^*(x, t), \psi^*(y, t)] = 0$$

and on $P^{\perp}\Gamma_s(\mathscr{H})$

$$[(\psi(x, t), \psi^*(y, t)] = \delta_y(x)I$$

Theorem 7.8. $i\hbar\,(\partial/\partial t)\psi(x, t) = H(x)(x, t)$.

Proof. Applying Corollary 7.2 gives

$$
\begin{aligned}
i\hbar \frac{\partial}{\partial t} \psi(x, t) &= [\psi(x, t), \Omega(H)] \\
&= e^{it\Omega(H)/\hbar}[\psi(x), \Omega(H)]e^{-it\Omega(H)/\hbar} \\
&= e^{it\Omega(H)/\hbar} H(x)\psi(x) e^{-it\Omega(H)/\hbar} \\
&= H(x)\psi(x, t) \quad \blacksquare
\end{aligned}
$$

8. KLEIN–GORDON FIELDS

This section treats a relativistic, free, scalar (spin-zero) field for a particle of mass m. Such a field would correspond to a spin-zero meson. The section illustrates how some of the heuristic manipulations of the standard theory can be made mathematically rigorous. We employ natural units in which $\hbar = c = 1$.

The one-particle Hilbert space $\mathscr{H}$ is taken to be the set of square-integrable complex functions on the mass hyperboloid with Lorentz-invariant measure. Thus, $\mathscr{H} = L^2(\mathbb{R}^3, d\mathbf{k}/k_0)$, where $d\mathbf{k} = dk_1\, dk_2\, dk_3$, $k_0 = (m^2 + \mathbf{k}^2)^{1/2}$, $\mathbf{k}^2 = k_1^2 + k_2^2 + k_3^2$, and the inner product is given by

$$
\langle \Phi, \Psi \rangle = \int d\mathbf{k} \frac{1}{k_0} \Phi^*(\mathbf{k}) \Psi(\mathbf{k})
$$

The free Hamiltonian H is defined by $(H\Phi)(\mathbf{k}) = k_0 \Phi(\mathbf{k})$ with domain

$$
\mathscr{D}(H) = \left\{ \Phi \in \mathscr{H} : \int d\mathbf{k}\, k_0 |\Phi(\mathbf{k})|^2 < \infty \right\}
$$

and the momentum operator $\mathbf{P} = (P_1, P_2, P_3)$ has the form $(\mathbf{P}\Phi)(\mathbf{k}) = \mathbf{k}\Phi(\mathbf{k})$ with domain

$$
\mathscr{D}(\mathbf{P}) = \left\{ \Phi \in \mathscr{H} \ \int d\mathbf{k} \frac{\mathbf{k}^2}{k_0} |\Phi(\mathbf{k})|^2 < \infty \right\}
$$

The energy-momentum operator $P = (H, \mathbf{P})$ is sometimes written P_μ, $\mu = 0, 1, 2, 3$. We also use the notation $k = (k_0, \mathbf{k})$, which is sometimes denoted k_μ, $\mu = 0, 1, 2, 3$. Using the notation

$$
\Box = \frac{\partial}{\partial x_0^2} - \nabla^2 = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2
$$

the Klein–Gordon equation becomes $(\Box + m^2)\eta(x) = 0$, where $x = (x_0, \mathbf{x})$ and x_0 corresponds to the time. Moreover, we shall need the Minkowski product

$$
k \cdot x = k_0 x_0 - \mathbf{k} \cdot \mathbf{x} = k_0 x_0 - k_1 x_1 - k_2 x_2 - k_3 x_3
$$

As before, the nonstandard Fock space is $\Gamma_s(\mathscr{H})$. Let $(f_i), j\in\mathbb{N}$, be an orthonormal basis of continuous functions for $\mathscr{H}$ and define $\psi_n(\mathbf{k}) = \sum_{j=0}^n f_j(\mathbf{k})a_n(f_j)$ and $\psi(\mathbf{k}) = [\psi_n(\mathbf{k})]$ as in Section 6. It follows from Theorem 7.3 that

$$\Omega(P_\mu) \approx \int d\mathbf{k}\,\frac{k_\mu}{k_0}\,\psi^*(\mathbf{k})\psi(\mathbf{k}) \tag{8.1}$$

and the particle number operator is

$$N = \Omega(I) \approx \int d\mathbf{k}\,\frac{1}{k_0}\,\psi^*(\mathbf{k})\psi(\mathbf{k}) \tag{8.2}$$

In the physics literature $\psi(\mathbf{k})$ and $\psi^*(\mathbf{k})$ are usually denoted $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^*$, respectively.

For $\Phi\in\mathscr{H}$, we define the "Fourier transform"

$$\hat{\Phi}(x) = \alpha\int d\mathbf{k}\,\frac{1}{k_0}\,e^{-ik\cdot x}\Phi(\mathbf{k})$$

where $\alpha = 1/\sqrt{2}(2\pi)^{3/2}$. More explicitly,

$$\hat{\Phi}(x_0,\mathbf{x}) = \alpha\int d\mathbf{k}\,\frac{1}{(m^2+\mathbf{k}^2)^{1/2}}\exp[-i(m^2+\mathbf{k}^2)^{1/2}x_0]\exp(-i\mathbf{k}\cdot\mathbf{x})\;\Phi(\mathbf{k})$$

It is easy to verify that $(\square^2+m^2)\hat{\Phi}(x)=0$, so $\hat{\Phi}$ satisfies the Klein–Gordon equation. Moreover, $\hat{\Phi}^*$ also satisfies the Klein–Gordon equation. We now define the operators $\phi_n^{(+)}(x)$ on $\mathscr{H}_s^n$ by

$$\phi_n^{(+)}(x) = \sum_{j=0}^n \hat{f}_j(x)a_n(f_j)$$

and we define

$$\phi_n^{(-)}(x) = (\phi_n^{(+)}(x))^* = \sum_{j=0}^n \hat{f}_j^*(x)a_n^*(f_j)$$

Since $\hat{f}_j$ and $\hat{f}_j^*$ satisfy the Klein–Gordon equation, $\phi_n^{(+)}$ and $\phi_n^{(-)}$ also satisfy this equation. As before, we define the field operators $\phi^{(+)}(x)$ and $\phi^{(-)}(x)$ on $\Gamma_s(\mathscr{H})$ by $\phi^{(+)}(x) = [\phi_n^{(+)}(x)]$ and $\phi^{(-)}(x) = [\phi_n^{(-)}(x)]$, which again satisfy the Klein–Gordon equation. Physically, $\phi^{(+)}(x)$ corresponds to the annihilation of a particle localized at x and $\phi^{(-)}$ corresponds to the creation of such a particle.

Lemma 8.1. The following equations hold:

$$\phi^{(+)}(x) = \alpha\int d\mathbf{k}\,\frac{1}{k_0}\,e^{-ik\cdot x}\psi(\mathbf{k})$$

$$\phi^{(-)}(x) = \alpha\int d\mathbf{k}\,\frac{1}{k_0}\,e^{ik\cdot x}\psi^*(\mathbf{k})$$

Proof. The result follows from

$$\phi_n^{(+)}(x) = \sum_{j=0}^{n} \alpha \int d\mathbf{k} \frac{1}{k_0} e^{-ik \cdot x} f_j(\mathbf{k}) a_n(f_j)$$
$$= \alpha \int d\mathbf{k} \frac{1}{k_0} e^{-ik \cdot x} \sum_{j=0}^{n} f_j(\mathbf{k}) a_n(f_j)$$
$$= \alpha \int d\mathbf{k} \frac{1}{k_0} e^{-ik \cdot x} \psi_n(\mathbf{k}) \quad \blacksquare$$

It is clear that

$$[\phi_n^{(+)}(x), \phi_n^{(+)}(y)] = [\phi_n^{(-)}(x), \phi_n^{(-)}(y)] = 0$$

and hence

$$[\phi^{(+)}(x), \phi^{(+)}(y)] = [\phi^{(-)}(x), \phi^{(-)}(y)] = 0$$

Moreover, applying (5.5), on the small space we have

$$[\phi_n^{(+)}(x), \phi_n^{(-)}(y)] = \sum_{i,j=0}^{n} \hat{f}_j(x) \hat{f}_j^*(y) [a_n(f_i), a_n^*(f_j)]$$
$$= \sum_{j=0}^{n} \hat{f}_j(x) \hat{f}_j^*(y) I$$

Hence,

$$[\phi^{(+)}(x), \phi^{(-)}(y)] = \left[\sum_{j=0}^{n} \hat{f}_j(x) \hat{f}_j^*(y) I \right]$$
$$= \alpha^2 \int d\mathbf{k} \frac{1}{k_0} e^{-ik \cdot x} \int d\mathbf{k}' \frac{1}{k_0} e^{ik' \cdot y} \left[\sum_{j=0}^{n} f_j(\mathbf{k}) f_j^*(\mathbf{k}') I \right]$$
$$= \alpha^2 \int d\mathbf{k} \frac{1}{k_0} e^{-ik \cdot x} \int d\mathbf{k}' \frac{1}{k_0} e^{ik' \cdot y} \delta_k(\mathbf{k}') I$$

The calculation so far is rigorous, but now in a heuristic sense this last expression is

$$\alpha^2 \int d\mathbf{k} \frac{1}{k_0} e^{-ik \cdot (x-y)} = i\Delta^{(+)}(x-y)$$

which is the usual form found in the literature.

We next discuss the Poincaré covariance of the field operators. To give a more relativistic notation, we identify a given $\Phi' \in \mathscr{H}$ with the function $\Phi(k) = \Phi'(\mathbf{k})$ where $k_0 = (m^2 + \mathbf{k}^2)^{1/2}$. The proper, orthochronous Poincaré group has a unitary representation $U(b, \Lambda)$ on $\mathscr{H}$ given by

$$[U(b, \Lambda)\Phi](k) = e^{ik \cdot b} \Phi(\Lambda^{-1} k)$$

We extend $U(b, \Lambda)$ to $\mathscr{H}_s^n$ by defining $U(b, \Lambda) = I$ on $\mathscr{H}^{(s)0}$ and for $\Phi \in \mathscr{H}^{(s)m}$, $1 \leq m \leq n$,

$$(U_n(b, \Lambda)\Phi)(k_1, \ldots, k_m) = e^{i\Sigma k_j \cdot b}\Phi(\Lambda^{-1}k_1, \ldots, \Lambda^{-1}k_m)$$

It is easy to verify that $U_n(b, \Lambda)$ is a unitary representation on $\mathscr{H}_s^n$ and hence $U(b, \Lambda) = [U_n(b, \Lambda)]$ is a unitary representation on $\Gamma_s(\mathscr{H})$.

Lemma 8.2. For $f \in \mathscr{H}$, we have

$$U_n(b, \Lambda)a_n(f)U_n(b, \Lambda)^* = a_n[U(b, \Lambda)f]$$

Proof. Let $\Phi \in \mathscr{H}^{(s)m}$, $1 \leq m \leq n$, and apply the Lorentz invariance of the measure to obtain

$$\begin{aligned}
&(U_n(b, \Lambda)a_n(f)\Phi)(k_1, \ldots, k_{m-1}) \\
&\quad = e^{i\Sigma k_j \cdot b}(a_n(f)\Phi)(\Lambda^{-1}k_1, \ldots, \Lambda^{-1}k_{m-1}) \\
&\quad = e^{i\Sigma k_j \cdot b}\sqrt{m}\int d\mathbf{k}\,\frac{1}{k_0}f^*(k)\Phi(k, \Lambda^{-1}k_1, \ldots, \Lambda^{-1}k_{m-1}) \\
&\quad = \sqrt{m}\int d\mathbf{k}\,\frac{1}{k_0}(e^{ik\cdot b}f(\Lambda^{-1}k))^* e^{ik\cdot b}e^{i\Sigma k_j \cdot b}\Phi(\Lambda^{-1}k, \Lambda^{-1}k_1, \ldots, \Lambda^{-1}k_{m-1}) \\
&\quad = (a_n[U(b, \Lambda)f]U_n(b, \Lambda)\Phi)(k_1, \ldots, k_{m-1})
\end{aligned}$$

Hence,

$$U_n(b, \Lambda)a_n(f) = a_n[U(b, \Lambda)f]U_n(b, \Lambda)$$

Multiply on the right by $U_n(b, \Lambda)^*$ to obtain the result. ∎

Applying Lemma 8.2, we obtain the covariance condition

$$U(b, \Lambda)a(f)U(b, \Lambda)^* = a[U(b, \Lambda)f]$$

We also have the following covariance conditions to within an infinitesimal.

Theorem 8.3. (a) $U(b, \Lambda)\psi(k)U(b, \Lambda)^* \approx e^{-i\Lambda k\cdot b}\psi(\Lambda k)$.
(b) $U(b, \Lambda)\phi^{(+)}(x)U(b, \Lambda)^* \approx \phi^{(+)}(\Lambda x + b)$.

Proof. (a) Applying Lemma 8.2, we have

$$U_n(b, \Lambda)\psi_n(k)U_n(b, \Lambda)^* = \sum_{j=0}^{n} f_j(k)a_n[U(b, \Lambda)f_j]$$

An argument similar to the proof of Lemma 7.5 gives

$$\left[\sum_{j=0}^{n} f_j(k)a_n(U(b, \Lambda)f_j)\right] \approx \left[\sum_{j=0}^{n} (U(b, \Lambda)^* f_j)(k)a_n(f_j)\right]$$

Since

$$U(b, \Lambda)^* = [U(b, \Lambda)]^{-1} = U((b, \Lambda)^{-1}) = U(-\Lambda^{-1}b, \Lambda^{-1})$$

we have

$$\begin{aligned} U(b, \Lambda)\psi(k)U(b, \Lambda)^* &\approx \left[\sum_{j=0}^{n} (U(-\Lambda^{-1}b, \Lambda^{-1})f_j)(k)a_n(f_j) \right] \\ &= \left[\sum_{j=0}^{n} e^{-ik \cdot \Lambda^{-1}b} f_j(\Lambda k)a_n(f_j) \right] \\ &= \left[e^{-i\Lambda k \cdot b} \sum_{j=0}^{n} f_j(\Lambda k)a_n(f_j) \right] \\ &= e^{i\Lambda k \cdot b}\psi(\Lambda k) \end{aligned}$$

(b) Applying the proof of Lemma 8.1, part (a), and the Lorentz invariance of the measures gives

$$\begin{aligned} U(b, \Lambda)\phi^{(+)}(x)U(b, \Lambda)^* &= \left[\alpha \int d\mathbf{k} \frac{1}{k_0} e^{-ik \cdot x} U_n(b, \Lambda)\psi_n(k)U_n(b, \Lambda)^* \right] \\ &\approx \left[\alpha \int d\mathbf{k} \frac{1}{k_0} e^{-ik \cdot x} e^{-ik \cdot \Lambda^{-1}b} \psi_n(\Lambda k) \right] \\ &= \left[\alpha \int d\mathbf{k} \frac{1}{k_0} e^{-ik \cdot (x + \Lambda^{-1}b)} \psi_n(\Lambda k) \right] \\ &= \left[\alpha \int d\mathbf{k} \frac{1}{k_0} e^{-i\Lambda k \cdot (\Lambda x + b)} \psi_n(\Lambda k) \right] \\ &= \left[\alpha \int d\mathbf{k} \frac{1}{k_0} e^{-ik \cdot (\Lambda x + b)} \psi_n(k) \right] \\ &= [\phi_n^{(+)}(\Lambda x + b)] = \phi^{(+)}(\Lambda x + b) \quad \blacksquare \end{aligned}$$

We now define the self-adjoint operators $\phi_n(x) = \phi_n^{(+)}(x) + \phi_n^{(-)}(x)$ on $\mathscr{H}_s^n$. Then the field operator

$$\phi(x) = [\phi_n(x)] = \phi^{(+)}(x) + \phi^{(-)}(x)$$

is a bounded internal self-adjoint operator on $\Gamma_s(\mathscr{H})$. It follows from Lemma 8.1 that

$$\phi_n(x) = \alpha \int d\mathbf{k} \frac{1}{k_0} (e^{-ik \cdot x}\psi_n(\mathbf{k}) + e^{ik \cdot x}\psi_n^*(\mathbf{k}))$$

and

$$\phi(x) = \alpha \int d\mathbf{k} \frac{1}{k_0} (e^{-ik \cdot x}\psi(\mathbf{k}) + e^{ik \cdot x}\psi^*(\mathbf{k}))$$

We define the *conjugate operators*

$$\pi_n(x) = \partial_0 \phi_n(x) = -i\alpha \int d\mathbf{k}\, (e^{-ik\cdot x}\psi_n(\mathbf{k}) - e^{ik\cdot x}\psi_n^*(\mathbf{k}))$$

$$\pi(x) = [\pi_n(x)] = -i\alpha \int d\mathbf{k}\, (e^{-ik\cdot x}\psi(\mathbf{k}) - e^{ik\cdot x}\psi^*(\mathbf{k}))$$

which are again self-adjoint. All the important operators can be expressed in terms of $\phi(x)$ and $\pi(x)$.

Lemma 8.4. The following formulas hold:

$$\psi(\mathbf{k}) = \alpha k_0 \int d\mathbf{x}\, \phi(x) e^{ik\cdot x} + i\alpha \int d\mathbf{x}\, \pi(x) e^{ik\cdot x}$$

$$\psi^*(\mathbf{k}) = \alpha k_0 \int d\mathbf{x}\, \phi(x) e^{-ik\cdot x} - i\alpha \int d\mathbf{x}\, \pi(x) e^{-ik\cdot x}$$

Proof. We can write $\phi_n(x)$ as follows:

$$\phi_n(x) = \alpha \int d\mathbf{k} \left(\frac{e^{-ik_0 x_0}\psi_n(\mathbf{k}) + e^{ik_0 x_0}\psi_n^*(-\mathbf{k})}{k_0} \right) e^{i\mathbf{k}\cdot\mathbf{x}}$$

Taking the Fourier transform gives

$$e^{-ik_0 x_0}\psi_n(\mathbf{k}) + e^{ik_0 x_0}\psi_n^*(-\mathbf{k}) = 2\alpha k_0 \int d\mathbf{x}\, \phi_n(x) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

Similarly,

$$\pi_n(x) = -i\alpha \int d\mathbf{k}\, (e^{-ik_0 x_0}\psi_n(\mathbf{k}) - e^{ik_0 x_0}\psi_n^*(-\mathbf{k})) e^{i\mathbf{k}\cdot\mathbf{x}}$$

and taking the Fourier transform gives

$$e^{-ik_0 x_0}\psi_n(\mathbf{k}) - e^{ik_0 x_0}\psi_n^*(-\mathbf{k}) = 2i\alpha \int d\mathbf{x}\, \pi_n(x) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

Adding these latter equations, we obtain

$$\psi_n(\mathbf{k}) = \alpha k_0 \int d\mathbf{x}\, \phi_n(x) e^{ik\cdot x} + i\alpha \int d\mathbf{x}\, \pi_n(x) e^{ik\cdot x}$$

The equation for $\psi(\mathbf{k})$ now follows and the equation for $\psi^*(\mathbf{k})$ follows by taking adjoints. ■

Substituting the expressions for $\psi(\mathbf{k})$ and $\psi^*(\mathbf{k})$ in Lemma 8.4 into (8.1) and (8.2) gives a representation of $\Omega(P_\mu)$ and N in terms of $\phi(x)$ and

$\pi(x)$. For example, using straightforward methods, it can be shown that

$$\Omega(H) \approx \frac{1}{2} \int d\mathbf{x}\, [\pi^2(x) + \nabla\phi(x) \cdot \nabla\phi(x) + m^2\phi^2(x)] + H_c$$

where

$$H_c = \left[i\alpha \int d\mathbf{k}\, k_0 \int d\mathbf{x} \int d\mathbf{x}'\, e^{ik\cdot(x-x')}[\phi_n(x), \pi_n(x')]_{x=x'_0} \right]$$

By the usual methods, the operator H_c can be eliminated by rewriting products in normal form.

We can also give a nonstandard treatment of charged scalar fields. Suppose we have oppositely charged particles of spin zero and mass m. Let the charges be $+e$ and $-e$ and call the first particles and the second antiparticles. Let $\mathscr{H} = L^2(\mathbb{R}^3, d\mathbf{k}/k_0)$ as before and define

$$\mathscr{H}_s^{n,n} = \mathscr{H}_s^n \otimes \mathscr{H}_s^n = \bigoplus_{j,j'=0}^{n} H^{(s)j} \otimes H^{(s)j'}$$

In this case, the first Hilbert space corresponds to particles and the second to antiparticles. We now form the nonstandard Fock space

$$\Gamma_s^2(\mathscr{H}) = \Gamma(\mathscr{H}_s^{n,n}: n \in \mathbb{N})$$

For $f \in \mathscr{H}$ we let $a(f)$ and $a^*(f)$ stand for $a(f) \otimes I$ and $a^*(f) \otimes I$, respectively, and define $b(f) = I \otimes a(f)$ and $b^*(f) = I \otimes a^*(f)$. We then define $\psi(\mathbf{k})$ just as before in terms of the a's and $\tilde{\psi}(\mathbf{k})$ analogously in terms of the b's. The particle number operators become

$$N_+ = \int d\mathbf{k}\, \frac{1}{k_0}\, \psi^*(\mathbf{k})\psi(\mathbf{k})$$

$$N_- = \int d\mathbf{k}\, \frac{1}{k_0}\, \tilde{\psi}(\mathbf{k})\tilde{\psi}(\mathbf{k})$$

and the *total charge operator* is defined to be the internal self-adjoint operator

$$Q = e(N_+ - N_-) = e \int d\mathbf{k}\, \frac{1}{k_0}\, (\psi^*(\mathbf{k})\psi(\mathbf{k}) - \tilde{\psi}^*(\mathbf{k})\tilde{\psi}(\mathbf{k}))$$

If A is a self-adjoint operator on $\mathscr{H}$, we can form the second quantization operator $\Omega(A)$ on $\Gamma_s^2(\mathscr{H})$ analogously to the way it was done for $\Gamma_s(\mathscr{H})$. The energy-momentum operator then becomes

$$\Omega(P_\mu) = \int d\mathbf{k}\, \frac{k_\mu}{k_0}\, (\psi^*(\mathbf{k})\psi(\mathbf{k}) + \tilde{\psi}^*(\mathbf{k})\tilde{\psi}(\mathbf{k}))$$

In the present context, the important field operators are

$$\phi^{(+)}(x) = \alpha \int d\mathbf{k} \frac{1}{k_0} \tilde{\psi}(\mathbf{k}) e^{-ik \cdot x}$$

$$\phi^{(-)}(x) = \alpha \int d\mathbf{k} \frac{1}{k_0} \psi^*(\mathbf{k}) e^{ik \cdot x}$$

These, as well as their adjoints, satisfy the Klein–Gordon equation. The field operator $\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$ corresponds to the creation of a charge e localized at x, and $\phi^*(x)$ corresponds to the annihilation of such a charge.

This section has just begun the study of a nonstandard quantum field theory. One can now proceed in a rigorous fashion to obtain other standard field-theoretic results.

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